STRANGE ATTRACTORS AND DYNAMICAL MODELS

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Three main types of strange attractors are described; namely hyperbolic, Lorenz-type and quasiattractors. In addition, a recent family of quasiattractors originating from Chua's circuit is briefly described. In connection with the strange attractors, we stress that models having quasiattractors containing structurally unstable homoclinic Poincaré orbits are "bad", in the sense of Ref. 24.

1. Introduction

One of the remarkable achievements in science in the 20th century is the discovery of dynamical chaos. Using this paradigm, many problems in modern science and engineering can be realistically simulated and analyzed via the tools of nonlinear dynamics. The quest for understanding and explaining chaos in real physical systems has led to the creation of new mathematical techniques. Indeed, there is historical precedence; namely, the quest for understanding oscillations in weakly nonlinear and quasilinear systems has led to Poincaré's theory of limit cycles and Lyapunov's stability theory.

However, many modern problems associated with systems involving high energies, powers, velocities, etc. must be modeled by multidimensional and strongly nonlinear differential equations (ordinary, partial etc.). The study of such systems has spawned numerous new concepts and terminologies: e.g. hyperbolic sets, symbolic dynamics, homoclinic and heteroclinic orbits, global bifurcations, strange attractors, entropy (topological and metric), Lyapunov's exponents, dimension, capacity, etc. In addition, dynamical chaos has also been characterized and studied by statistical methods, including an extensive application of numerical and experimental analysis of correlation functions and power spectrums.

In the creation of new mathematical techniques for studying dynamical chaos, the methods of qualitative theory, bifurcation theory and the theory of strange attractors have played a particularly important role. In particular, strange attractors have appeared as a mathematical image of dynamical chaos.

Strange attractors in finite-dimensional dynamical systems can be divided into three main classes: hyperbolic, Lorenz-type and quasiattractors (abbreviation of quasistochastic attractors). In this paper, using the recent results of "modules" and "good or bad" systems we discuss their main individual properties, similarities, and peculiarities in order to investigate the possibility of making a *complete* bifurcation analysis of a given dynamical system.

We will discuss briefly also a recent family of strange attractors; namely, Chua's attractors, which have been the object of numerous recent research papers on Chua's circuit family. The main motivation for studying strange attractors from Chua's circuit is that these attractors can be easily observed from a remarkably simple electrical circuit made of only 4 linear circuit elements (1 resistor, 1 inductor, and 2 capacitors) and a single nonlinear element (Chua's diode) characterized by a function of only one variable $f(x)$. Moreover, it can be verified that Chua's circuit is the *simplest* among all autonomous electronic circuits that can become chaotic.[1]

2. Hyperbolic Attractors

Hyperbolic attractors are the limiting sets for which the axiom A of Smale is valid, and hence, they are structurally stable. For these attractors, periodic orbits as well as homoclinic orbits are everywhere dense. For hyperbolic attractors all trajectories are of the same saddle type, i.e. the stable (respectively, unstable) manifold of all trajectories have the same dimension. Examples of these attractors include Anosov's systems, Smale–Williams' solenoid, Plykin's attractors, etc. However, the main disadvantage of hyperbolic attractors is that their realizations in the form of maps or differential equations, let alone applications, have so far not been found.

3. Lorenz-Type Attractors

A peculiarity of Lorenz-type attractors is their similarity to the hyperbolic ones; namely, their structurally stable (hyperbolic) periodic orbits and their homoclinic and heteroclinic orbits etc. are everywhere dense, and under small perturbations no stable orbits arise. Nevertheless Lorenz-type attractors are not structurally stable. The cause of this property is due to the imbedding of a saddle-type equilibrium point having a one-dimensional unstable manifold in the attractor.

The Lorenz-type attractor was first observed in the Lorenz equation[2]

$$\begin{aligned} \dot{x} &= -\sigma(x-y) \\ \dot{y} &= -y + rx - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{1}$$

which is derived from a Galerkin's approximation of the partial differential equations describing the convection of fluid flows, or from the dynamics of the simplest laser model.

Two geometric models have been used to study Lorenz-type attractors:

(1) Williams'[3] model which represents the inverse limit of a semiflow on a branching two-dimensional manifold.

(2) Afraimovich, Bykov and Shil'nikov's[4,5] model which is represented by a two-dimensional discontinuous map T, which is smooth outside of the line of discontinuity. The map T is the Poincaré map on the same global section D, transversal to the stable manifold of the saddle point (Fig. 1).

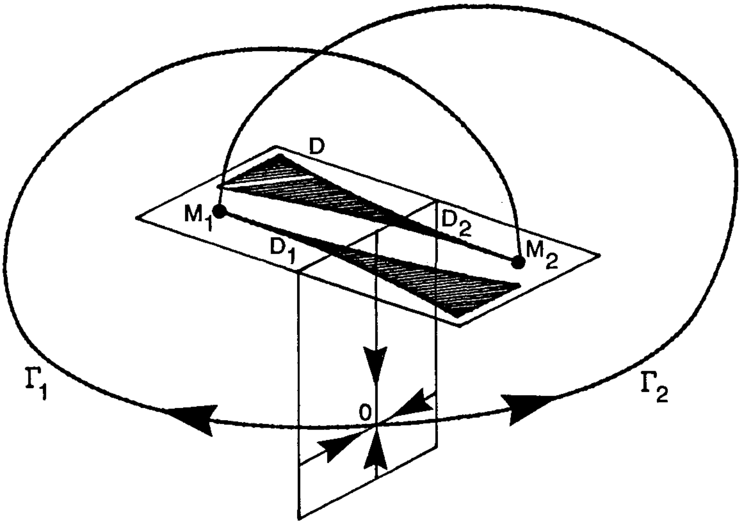

Fig. 1. A two-dimensional map T. Dashed triangles are the images of D_1 and D_2 (see the text). Γ_1 and Γ_2 are the separatrices of the saddle point 0.

Here the dashed triangles are the images under the map T of the area $D = [-\alpha \leq x \leq \alpha,\ |y| < \beta]$; or more precisely, of the two connected parts $D_1 = [|x| < \alpha,\ 0 < y < \beta]$ and $D_2 = [|x| < \alpha,\ -\beta < y < 0]$; Γ_1 and Γ_2 are the trajectories going out of the saddle point O (separatices).

The mapping T can be written in the form

$$\begin{aligned} \bar{x} &= f(x,y) \\ \bar{y} &= g(x,y) \end{aligned} \tag{2}$$

where $(x,y) \in D_1 \cup D_2$ and the functions $f \in \mathbb{C}^2$, $g \in \mathbb{C}^2$ satisfy the following conditions:

$$\begin{aligned} &\|(\partial f/\partial x)\| < 1\,, \quad \|(\partial g/\partial y)^{-1}\| < 1 \\ &\|(\partial g/\partial y)^{-1}(\partial f/\partial y)\| \cdot \|(\partial g/\partial x)\| < (1 - \|(\partial g/\partial y)^{-1}\|) \cdot (1 - \|(\partial f/\partial x)\| \\ &1 - \|(\partial g/\partial y)^{-1}\| \cdot \|(\partial f/\partial x)\| \\ &\qquad > 2\sqrt{\|(\partial g/\partial y)^{-1}\| \cdot \|(\partial g/\partial x)\| \cdot \|(\partial g/\partial y)^{-1}(\partial f/\partial y)\|} \end{aligned} \tag{3}$$

where $\|\cdot\| = \sup_{(x,y)\in D_1 \cup D_2} |\cdot|$.

It is assumed that (f,g) under $y \to 0\,(y > 0)$ is defined by the point M_1 and under $y \to 0\,(y < 0)$ by the point M_2. Here $y = 0$ is a discontinuous gap line.

The geometrical meaning of the above conditions is that there must be a contraction in the x-direction and an expansion in the y-direction under the action of T. Therefore, T cannot have more than one fixed point of the saddle type in D_i, $i = 1, 2$, where D_i has the shape of a triangle (Fig. 1).

This type of mappings gives rise to the Lorenz-type attractors, which have been proved to be structurally unstable. Indeed, they are the basic bifurcations which give rise to lacunas and unstable limiting sets related to them.

By virtue of this theory the bifurcations which give rise to attractors[4] were derived and the parameter regions corresponding to the existence of both symmetric and asymmetric Lorenz-type attractors,[6,7] the Shimizu-Morioka attractor[8–10] as well as the above conditions were verified numerically. Hence, criteria which give conditions on the existence of Lorenz-type attractors for an arbitrary vector fields are of great interest. Some of these conditions were obtained by the author and were first presented at the combined Petrovsky's Seminar and the Moscow Mathematical Society Meeting.[11] Since these conditions turned out to be quite effectively and easily checked for certain systems, they will be discussed below.

Consider a finite-number parameter family of vector fields defined by the system of equations

$$\dot{x} = X(x, \mu) \tag{4}$$

where $x \in \mathbb{R}^{n+1}$, $\mu \in \mathbb{R}^m$, and $X(x, \mu)$ represents a C^r-smooth function of x and μ.

Assume the following conditions are satisfied:

(1) The system (4) has a saddle-type equilibrium point $O(0,0)$. The eigenvalues of the Jacobian matrix $DX(0,0)$ satisfies the conditions:

$$\begin{aligned} &\operatorname{Re} \lambda_i < 0\,, \quad i = 1, \ldots, n\,, \quad \lambda_0 \text{ is real and } \lambda_0 > 0\,; \\ &\lambda_1 \text{ is real and } \operatorname{Re} \lambda_i < \lambda_1\,, \quad i = 2, \ldots, n\,. \end{aligned}$$

(2) Trajectories Γ_1 and Γ_2, originating from 0 at $t = 0$, eventually return to 0, and tangent to each other as $t \to \infty$, i.e. $\bar{\Gamma}_1 \cup \bar{\Gamma}_2$ forms a "figure eight" butterfly (Fig. 2).

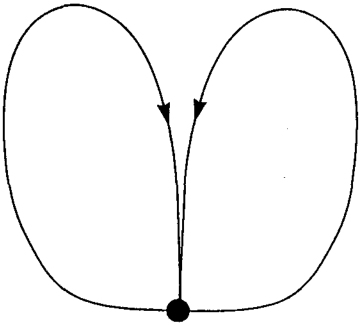

Fig. 2. Trajectories Γ_1 (left) and Γ_2 (right) form a "figure eight" butterfly.

Then, for $\mu > 0$ in the parameter space there exists an open set V whose boundary contains the origin such that the system (4) has a Lorenz-type attractor for the following cases:

Case 1:

(1) Γ_1 and Γ_2 tend to 0, tangential to each other along a dominating direction defined by the eigenvector of the eigenvalue λ_1.

(2) $1/2 < \gamma < 1$, $\nu_i > 1$, $\gamma = -\lambda_1 \lambda_0^{-1}$, $\nu_i = \operatorname{Re} \lambda_i \cdot \lambda_0^{-1}$.

(3) The separatix's values A_1 and A_2 (defined in Ref. 5) are equal to zero.

Here we may consider the dimension m of the controlling parameters to be equal to 4; μ_1 and μ_2 control the behavior of Γ_1 and Γ_2 respectively, but μ_3 and μ_4 can be chosen equal to A_1 and A_2.

Case 2:

(1) $1/2 < \gamma < 1$, $\nu_i > 1$

(2) λ_2 is real and Re $\lambda_i < \lambda_2$, $i = 3, \ldots, n$

(3) G_1 and G_2 belong to the nondominating manifold $W_0^{ss} \subset W_0^s$ (i.e. to the eigenspace of λ_i's for $i = 3, \ldots, n$) and tend to 0, tangential to each other along a dominating direction defined by the eigenvector of the eigenvalue λ_1.

Here we may consider $m = 4$; μ_1 and μ_2 control Γ_1 and Γ_2, but μ_3 and μ_4 characterize the distance between Γ_1 and Γ_2 respectively and at $\mu_1 = \mu_2 = 0$, the distance between Γ_1 and W_0^{ss}.

Case 3:

(1) $G_i \notin W_0^{ss}$, $i = 1, 2$

(2) $\gamma = 1$

(3) $A_i \neq 0$ and $|A_i| < 2$, $i = 1, 2$.

Here we may consider $m = 3$, μ_1 and μ_2 control Γ_1 and Γ_2, but $\mu_3 = \gamma - 1$.

In the symmetrical case all these bifurcations have codimension two. Note that in each of the cases 1, 2, or 3 it is necessary to study the subclasses defined by the following conditions:

(A) orientable $A_1 > 0$, $A_2 > 0$

(B) semiorientable $A_1 > 0$, $A_2 < 0$ and

(C) nonorientable $A_1 < 0$, $A_2 < 0$.

It was shown in Refs. 8 and 10 that both subclasses (A) and (C) are realized in the Shimizu–Morioka model, i.e. this model has both orientable and nonorientable Lorenz-type attractors. Using the results of Ref. 5, the birth of the lacunas and the death of the attractors were explained. The subclass (A) was studied also by Rychlic[12] and the subclass (C) by Robinson.[13,a]

The theory of invariant measures (Sinai–Bowen–Ruelle measure) was applied to the Lorenz-type attractors and the results are similar to those characterizing

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[a] We remark that the Belykh's[14] attractor of a two-dimensional discontinuous map may also be considered as a Lorenz-type attractor. This attractor can be classified somewhere in between the hyperbolic and the Lorenz-type attractors.

hyperbolic attractors. By virtue of this, Sinai introduced the following notion of a stochastic attractor[15]:

A stochastic attractor is an invariant closed set A in the phase space with the following properties:

(1) There exists a neighborhood U, $A \subset U$, such that if $x \in U$, then $\operatorname{dist}(x(t), A) \longrightarrow 0$, as $t \longrightarrow +\infty$.
(2) For any initial probability distribution P_0 on A, its shift as $t \longrightarrow \infty$ converges to an invariant distribution P on A, independently of P_0.
(3) The probability distribution P is mixing, i.e. the autocorrelation function tends to zero as $t \longrightarrow \infty$.

Note that the mixing condition excludes the existence of stable orbits.

Both hyperbolic and Lorenz-type attractors are stochastic attractors and hence the ergodic theory[15–17,28] can be used for their characterization. Small random perturbations essentially do not influence these attractors because the dynamic stochasticity tends to dominate the white noise.

The problem of the next types of strange attractors (called quasiattractors) is more complicated, at least for three-dimensional systems, because they are not stochastic. Hence, we believe that the effects due to small perturbations must be included in the stochastic analysis of quasiattractors.

4. Quasiattractors

The term quasiattractor[20] denotes the limiting set enclosing the periodic orbits of different topological types, structurally unstable homoclinic Poincaré trajectories, which may not be transitive, etc. First, note that these attractors have not been sufficiently studied, particularly in the case of hyperchaos. One of the main reasons for the complexity of quasiattractors is the existence of a structurally unstable homoclinic orbit of either the system (4), or of a system "close" to it. According to Refs. 21–24, this implies the sensitive dependence of the attractor structure on any small variation on the right-hand side of (4). Thus, for three-dimensional systems, besides the nontrivial hyperbolic set of trajectories, a countable set of stable orbits may coexist. Such behavior has been observed in many models, including the Lorenz system[6] (at $\sigma = 10$, $b = 8/3$, $\gamma > 31$), the Henon map, systems with spiral attractor chaos,[33] etc. This kind of attractors is the most important because of its wide applications[29,30] (see also references in Ref. 31).

In recent years, another family of attractors, called Chua's attractors, has been the subject of many research papers. One reason for such intense research activities on Chua's circuit is that, for a certain parameter range (α, β, m_0, m_1), the associated autonomous system

$$\begin{aligned} \dot{x} &= \alpha(y - h(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y \end{aligned} \tag{5}$$

where $h(x) = m_1x + (m_0 - m_1)[|x+1| - |x-1|]/2$, satisfies the conditions of Ref. 32, thereby proving rigorously that (5) is chaotic in the sense of Refs. 32 and 33. Two members of the family of Chua's attractors – the spiral-type and the double-scroll Chua's attractors – may be classified as quasiattractors over a certain region of the parameter space.

The analytical difficulties in our study of quasiattractors suggest an important question: is it generally possible to carry out a complete investigation of an arbitrary finite-number parameter family of differential equations?

This question is related to the following two fundamental problems, in the qualitative study of differential equations first posed by Andronov: (1) to split the parameter space into domains of structural stability and to determine the bifurcation set (2) to separate the bifurcation set into connected components having the same phase portrait (in the sense of topological equivalence).

In view of the above problems, a "good" model must have a sufficient number of parameters for uncovering all possible types of bifurcation phenomena of equilibrium points, periodic orbits, homoclinic and heteroclinic trajectories, etc. The formal notion of a "good" model is given in Ref. 24. The essence of it is that a good model has a sufficient number of parameters for determining all bifurcations such that "nearby" models are equivalent.

It is natural to pose our problem of complete qualitative investigation for a good model. If a model is "bad" we must first increase the number of parameters, and if this fails, we must redefine the equivalence relation.

Obviously, all bifurcations in a good model have a finite codimension and the model is transversal to any bifurcation submanifold in the Banach space of dynamical systems. In the case of differential equations on a plane it seems that these necessary conditions are also sufficient for the model's "goodness" (at least it is so in the case of small codimensions). Unfortunately, for the multidimensional case basic differences arise even for the class of systems having simple dynamics, e.g. systems without homoclinic Poincaré orbits. For example, in order for a family of ODE in $\mathbb{R}^n$, $n > 2$, to be good it is necessary for it to have a finite-number of "modules", which is defined as follows:

Definition: We say that a system X has a *module*, if X belongs to a Banach manifold M in which a continuous and locally nonconstant function $h(\cdot)$ is defined such that for any equivalent X_1, $X_2 \in M$ the conjugacy relation $h(X_1) = h(X_2)$ holds.

We say that X has m modules if X belongs to a Banach manifold at which m independent modules are defined. We say X has a countable number of modules if any given finite-number of modules can be defined on the manifold.

Modules for systems with simple dynamics were discovered by Palis[25] (for the case of diffeomorphism $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ having structurally unstable heteroclinic orbits). The conditions providing the countable number of modules were obtained in Ref. 26. However, note that in the case of Ω-equivalence the systems from Refs. 25 and 26 are Ω-structurally stable and any finite-number parameter family is good in this sense.

The situation is more complicated for systems having strange attractors since the structurally unstable systems may fill out entire regions. This type of systems includes those models having Lorenz-type attractors, and models having structurally unstable homoclinic Poincaré orbits (Fig. 3).

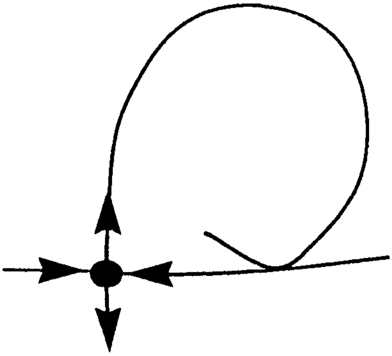

Fig. 3. A structurally unstable homoclinic orbit.

In the case of the Lorenz-type attractors a pair of kneading-invariants is the full invariant. Therefore one can study the dynamics completely via a two-parameter model. In the second case the situation is very complicated. In particular, it is related to the following phenomenon discovered by Newhouse[22]: any model having a homoclinic orbit which is transversal to a codimension-one bifurcation set (a model such that systems having quadratic-tangent stable and unstable manifolds are everywhere dense), must intersect some domain in the Banach space in which systems with structurally unstable homoclinic orbits in this domain are everywhere dense. Here, the inequality $|\lambda\gamma| < 1$ is assumed to hold (λ, γ are characteristic multipliers of the periodic orbits).

The next result is given in Ref. 24.

The following C^r-smooth ($r \geq 3$) systems are everywhere dense in the Newhouse domains:

(1) Systems with a countable set of Ω-equivalent modules.

(2) Systems having periodic motions with any order of degeneracy for characteristic multipliers equal to 1 and -1.

(3) Systems having homoclinic orbits which are tangent at the intersection of the stable and unstable invariant manifolds of any order.

We have shown that models having structurally unstable Poincaré homoclinic orbits cannot be good in the sense of Ω-equivalence. Consequently, we can make the following important conclusion: the problem of *complete* investigation of the three-dimensional systems having quasiattractors is unrealistic. A precise statement of this problem turns out to be very complicated. It appears that one should refrain from the fruitless ideology of "complete description" and turn to the study of some special but typical properties of the system. In this case, which properties are of main interest depend on the nature of the problem.

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