International Journal of Bifurcation and Chaos, Vol. 24, No. 8 (2014) 1440003 (10 pages)

DOI: 10.1142/S0218127414400033

Showcase of Blue Sky Catastrophes*

Leonid Pavlovich Shilnikov
Research Institute for Applied Mathematics and Cybernetics,
Nizhny Novgorod, 603005, Russia

Andrey L. Shilnikov
Neuroscience Institute and Department of Mathematics and Statistics,
Georgia State University, Atlanta 30303, USA
Department of Computational Mathematics and Cybernetics,
Lobachevsky State University of Nizhny Novgorod,
Nizhny Novgorod, 603950, Russia
ashilnikov@gsu.edu

Dmitry V. Turaev
Department of Mathematics, Imperial College,
London, SW7 2AZ, UK
dturaev@imperial.ac.uk

Received February 2, 2014

Let a system of differential equations possess a saddle periodic orbit such that every orbit in its unstable manifold is homoclinic, i.e. the unstable manifold is a subset of the (global) stable manifold. We study several bifurcation cases of the breakdown of such a homoclinic connection that causes the blue sky catastrophe, as well as the onset of complex dynamics. The birth of an invariant torus and a Klein bottle is also described.

Keywords: Codimension-one; homoclinic bifurcations; blue sky catastrophe.

1. Introduction

In the pioneering works by Andronov and Leontovich [Andronov & Leontovich, 1937; Andronov *et al.*, 1971] all main bifurcations of stable periodic orbits of dynamical systems in a plane had been studied: the emergence of a limit cycle from a weak focus, the saddle-node bifurcation through a merger of a stable limit cycle with an unstable one and their consecutive annihilation, the birth of a limit cycle from a separatrix loop to a saddle, as well as from a separatrix loop to a saddle-node equilibrium. Later, in the 50–60s these bifurcations were generalized for the multidimensional case, along with two additional bifurcations: period doubling and the birth of a two-dimensional torus. Apart from that, Shilnikov [1966, 1968] had studied the main bifurcations of

*The paper was written more than three years ago. In 2010, L. P. Shilnikov worked on several papers, remaining still unpublished. His idea was to launch a new direction in the theory of homoclinic bifurcations — the study of codimension-∞ bifurcations at a perturbation of nontrivial configurations of unstable manifolds. This paper was meant to be the first in the series; in the Summer of 2011 it was submitted to a collection volume on the occasion of V. Afraimovich's 65th anniversary. Because of an unfortunate delay in the release of the volume, we, A. Shilnikov and D. Turaev, decided to present the text here.

saddle periodic orbits out of homoclinic loops to a saddle and discovered a novel bifurcation of homoclinic loops to a saddle–saddle.[1]

Nevertheless, an open problem still remained: could there be other types of codimension-one bifurcations of periodic orbits? Clearly, the emphasis was put on bifurcations of *stable* periodic orbits, as only they generate robust self-sustained periodic oscillations, the original paradigm of nonlinear dynamics. One can pose the problem as follows:

> *In a one-parameter family X_μ of systems of differential equations, can both the period and the length of a structurally stable periodic orbit $\mathcal{L}_\mu$ tend to infinity as the parameter μ approaches some bifurcation value, say $\mu_0 = 0$?*

Here, structural stability means that none of the multipliers of the periodic orbit $\mathcal{L}_\mu$ crosses the unit circle, i.e. $\mathcal{L}_\mu$ does not bifurcate at $\mu \neq \mu_0$. Of particular interest is the case where $\mathcal{L}_\mu$ is stable, i.e. all the multipliers are strictly inside the unit circle.

A similar formulation was given by Palis and Pugh [1975] (notable Problem 37), however the structural stability requirement was missing there. Exemplary bifurcations of a periodic orbit whose period becomes arbitrarily large while the length remains finite as the bifurcation moment is approached are a homoclinic bifurcation of a saddle with a negative saddle value and that of a saddle-node [Shilnikov, 1963; Shilnikov *et al.*, 2001]. These were well-known at the time, so in [Palis & Pugh, 1975] an additional condition was imposed, in order to ensure that the sought bifurcation is really of a new type: the periodic orbit $\mathcal{L}_\mu$ must stay away from any equilibrium states (this would immediately imply that the length of the orbit grows to infinity in proportion to the period). As Abraham put it, the periodic orbit must "disappear in the blue sky" [Abraham, 1985].

In fact, a positive answer to "Problem 37" could be found in an earlier paper [Fuller, 1967]. In explicit form, a solution was proposed by Medvedev [1982]. He constructed examples of flows on a torus and a Klein bottle with stable limit cycles whose lengths and periods tend to infinity as $\mu \to \mu_0$, while at $\mu = \mu_0$ both the periodic orbits disappear and new, structurally unstable saddle-node periodic orbits appear (at least two of them, if the flow is on a torus). The third example of [Medvedev, 1982] was a flow on a three-dimensional torus whose orbits are all periodic and degenerate, and for the limit system the torus is foliated by two-dimensional invariant tori.

Medvedev's examples are not of codimension-one: this is obvious for the torus case that requires at least two saddle-nodes, i.e. X_{μ_0} is of codimension-two at least. In case of the Klein bottle one may show [Shilnikov *et al.*, 2001; Afraimovich & Shilnikov, 1982; Shilnikov & Turaev, 2000; Li & Zhang, 1991; Ilyashenko & Li, 1999] that for a generic perturbation of the Medvedev family the periodic orbits existing at $\mu \neq \mu_0$ will not remain stable for all μ as they undergo an infinite sequence of forward and backward period-doubling bifurcations (this is a typical behavior of fixed points of a nonorientable diffeomorphism of a circle).

A blue sky catastrophe of codimension-one was found only in 1995 by Shilnikov and Turaev [Shilnikov & Turaev, 2000; Turaev & Shilnikov, 1995; Shilnikov & Turaev, 1997, 2006]. The solution was based on the study of bifurcations of a saddle-node periodic orbit whose entire unstable manifold is homoclinic to it. The study of this bifurcation was initiated by Afraimovich and Shilnikov [Afraimovich & Shilnikov, 1982, 1974a, 1974b, 1978] for the case where the unstable manifold of the saddle-node is a torus or a Klein bottle (see Fig. 1). As soon as the saddle-node disappears, the Klein bottle may persist, or it may break down to cause chaotic dynamics in the system [Afraimovich & Shilnikov, 1991; Newhouse *et al.*, 1983; Turaev & Shilnikov, 1986; Shilnikov *et al.*, 2004]. In these works, most of the attention was paid to the torus case, as its breakdown provides a geometrical model of the quasiperiodicity-toward-chaos transition encountered universally in Nonlinear Dynamics, including the onset of turbulence [Shilnikov, 1991].

In the hunt for the blue sky catastrophe, other distinct configurations of the unstable manifold of the saddle-node were suggested in [Turaev & Shilnikov, 1995]. In particular, it was shown that in the phase space of dimension 3 and higher the

[1]An equilibrium state, alternatively called a Shilnikov saddle-node, due to a merger of two saddles of different topological types.

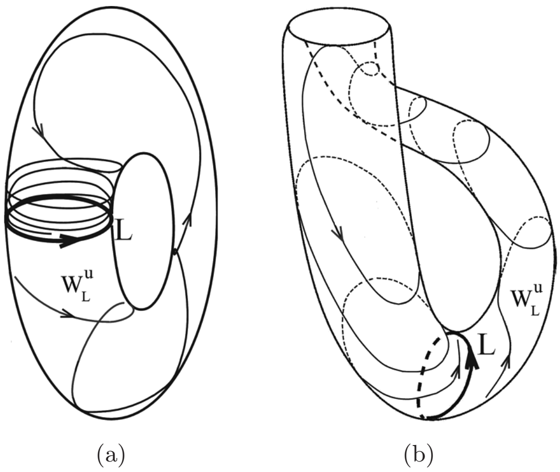

Fig. 1. Two cases of the unstable manifold W_L^u homoclinic to the saddle-node periodic orbit L: (a) a 2D torus or (b) a Klein bottle.

homoclinic trajectories may spiral back onto the saddle-node orbit as shown in Fig. 2. If we have a one-parameter family X_μ of systems of differential equations with a saddle-node periodic orbit at $\mu = \mu_0$ which possesses this special kind of homoclinic unstable manifold and satisfy certain additional conditions, then as the saddle-node disappears the inheriting attractor consists of a single stable periodic orbit $\mathcal{L}_\mu$ which undergoes no bifurcation as $\mu \to \mu_0$ while its length tends to infinity. Its topological limit, M_0, is the entire unstable manifold of the saddle-node periodic orbit.

The conditions found in [Turaev & Shilnikov, 1995] for the behavior of the homoclinic orbits

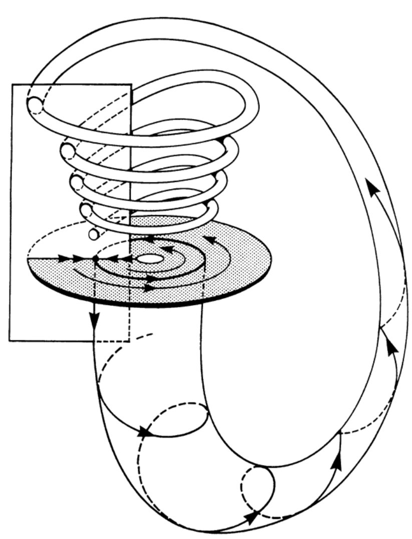

Fig. 2. Original construction of the blue sky catastrophe from [Turaev & Shilnikov, 1995].

ensuring the blue sky catastrophe are open, i.e. a small perturbation of the one-parameter family X_μ does not destroy the construction. This implies that such a blue sky catastrophe occurs any time a family of systems of differential equations crosses the corresponding codimension-one surface in the Banach space of smooth dynamical systems. This surface constitutes a stability boundary for periodic orbits. This boundary is drastically new comparable to those known since the 30–60s and has no analogues in planar systems. There are reasons to conjecture that this type of the blue sky catastrophe closes the list of main stability boundaries for periodic orbits (i.e. any new stability boundary will be of codimension greater than one).

In addition, another version of blue sky catastrophe leading to the birth of a uniformly-hyperbolic strange attractor (the Smale–Williams solenoid [Smale, 1967; Williams, 1974]) was also discovered in [Turaev & Shilnikov, 1995; Shilnikov & Turaev, 1997]. This codimension-one bifurcation of a saddle-node corresponds yet to a different configuration of the homoclinic unstable manifold of the periodic orbit (the full classification is presented in [Shilnikov *et al.*, 2001]). Here, the structurally stable attractor existing all the way up to $\mu = \mu_0$ does not bifurcate so that the length of each and every (saddle) periodic orbit in it tends to infinity as $\mu \to \mu_0$.

Initially we believed that the corresponding configuration of the unstable manifold would be too exotic for the blue sky catastrophe to occur naturally in a plausible system. In contrast, soon after, a first explicit example of the codimension-one blue sky catastrophe was proposed by Gavrilov and Shilnikov [2000], in the form of a family of 3D systems of differential equations with polynomial right-hand sides. A real breakthrough came in when the blue sky catastrophe turned out to be a typical phenomenon for slow–fast systems. Namely, in [Shilnikov *et al.*, 2001; Shilnikov *et al.*, 2005] we described a number of very general scenarios leading to the blue sky catastrophe in such systems with at least two fast variables; for systems with one fast variable the blue sky catastrophe was found in [Glyzin *et al.*, 2008]. In this way, the blue sky catastrophe has found numerous applications in mathematical neuroscience, namely, it explains a smooth and reversible transition between tonic spiking and bursting in exact Hodgkin–Huxley type models of interneurons [Shilnikov & Cymbalyuk, 2005, 2004] and in mathematical models of square-wave bursters [Shilnikov & Kolomiets, 2008]. The great variability of the burst duration near the blue sky catastrophe was shown to be the key mechanism ensuring the diversity of rhythmic patterns generated by small neuron complexes that control invertebrate locomotion [Belykh & Shilnikov, 2008; Belykh *et al.*, 2010; Wojcik *et al.*, 2011; Wojcik *et al.*, 2014].

In fact, the term "blue sky catastrophe" should be naturally treated in a broader way. Namely, under this term we allow to embrace a whole class of dynamical phenomena that all are due to the existence of a stable (or, more generally, structurally stable) periodic orbit, $\mathcal{L}_\mu$, depending continuously on the parameter μ so that both, the length and the period of $\mathcal{L}_\mu$ tend to infinity as the bifurcation parameter value is reached. As for the topological limit, M_0, of the orbit $\mathcal{L}_\mu$ is concerned, it may possess a rather degenerate structure that does not prohibit M_0 from having equilibrium states included. As such, the periodic regime $\mathcal{L}_\mu$ could emerge as a composite construction made transiently of several quasi-stationary states: nearly constant, periodic, quasiperiodic, and even chaotic fragments. As one of the motivations (which we do not pursue here) one may think of slow–fast model where the fast 3D dynamics is driven by a periodic motion in a slow subsystem.

2. Results

In this paper, we focus on an infinitely degenerate case where M_0 is comprised of a saddle periodic orbit with a continuum of homoclinic trajectories. Namely, we consider a one-parameter family of sufficiently smooth systems of differential equations X_μ defined in R^{n+1}, $n \geq 2$, for which we need to make a number of assumptions as follows.

(A) There exists a saddle periodic orbit L (we assume the period equals 2π) with the multipliers[2] $\rho_1, \ldots, \rho_n$. Let the multipliers satisfy

$$\max_{i=2,\ldots,n-1} |\rho_i| < |\rho_1| < 1 < |\rho_n|. \qquad (1)$$

Once this property is fulfilled at $\mu = 0$, it implies that the saddle periodic orbit $L = L_\mu$ exists for all small μ and smoothly depends on μ. Condition (1) also holds for all small μ. This condition implies that the stable manifold W^s_μ is n-dimensional[3] and the unstable manifold W^u_μ is two-dimensional. If the unstable multiplier ρ_n is positive (i.e. $\rho_n > 1$), then the orbit L_μ divides W^u_μ into two halves, W^+_μ and W^-_μ, so $W^u_\mu = L_\mu \cup W^+_\mu \cup W^-_\mu$. If ρ_n is negative ($\rho_n < -1$), then W^u_μ is a Möbius strip, so L_μ does not divide W^u_μ; in this case we denote $W^+_\mu = W^u_\mu \backslash L_\mu$.

Concerning the stable manifold, condition (1) implies that in W^s_μ there exists (at $n \geq 3$) an $(n-1)$-dimensional strong-stable invariant manifold W^{ss}_μ whose tangent at the points of L_μ contains the eigen-directions corresponding to the multipliers $\rho_2, \ldots, \rho_{n-1}$, and the orbits in $W^s_\mu \backslash W^{ss}_\mu$ tend to L_μ along the direction which correspond to the leading multiplier ρ_1.

(B) At $\mu = 0$ we have $W^+_0 \subset W^s_0 \backslash W^{ss}_0$, i.e. we assume that *all* orbits from W^+_0 are homoclinic to L. Moreover, as $t \to +\infty$, they tend to L along the leading direction.

(C) We assume that the flow near L contracts three-dimensional volumes, i.e.

$$|\rho_1 \rho_n| < 1. \qquad (2)$$

This condition is crucial, as the objects that we obtain by bifurcations of the homoclinic surface $W^+_0 \cup L$ are meant to be attractors. Note that this condition is similar to the negativity of the saddle value condition from the theory of homoclinic loops to a saddle equilibrium [Andronov & Leontovich, 1937; Andronov *et al.*, 1971; Shilnikov, 1963], see (6).

(D) We assume that one can introduce linearizing coordinates near L. Namely, a small neighborhood U of L is a solid torus homeomorphic to $S^1 \times R^n$, i.e. we can coordinatize it by an angular variable θ and by normal coordinates $u \in R^n$. Our assumption is that these coordinates are chosen so that the system in the small neighborhood of L takes the form

$$\dot{u} = C(\theta, \mu)u, \quad \dot{\theta} = 1, \qquad (3)$$

where C is 2π-periodic in θ. The smooth linearization is not always possible, and our results can be obtained without this assumption. We, however,

[2] The eigenvalues of the linearization of the Poincaré map.

[3] The intersection of W^s_μ with any cross-section to L_μ is $(n-1)$-dimensional.

will avoid discussing the general case here, in order to make the construction more transparent.

It is well-known that by a 4π-periodic transformation of the coordinates u, system (3) can be brought to the time-independent form. Namely, we may write the system as follows

$$\begin{aligned}&\dot{x} = -\lambda(\mu)x, \quad \dot{y} = B(\mu)y,\\&\dot{z} = \gamma(\mu)z,\\&\dot{\theta} = 1,\end{aligned} \tag{4}$$

where $x \in R^1$, $y \in R^{n-2}$, $z \in R^1$, and $\lambda = -\frac{1}{2\pi}\ln|\rho_1| > 0$, $\gamma = \frac{1}{2\pi}\ln|\rho_n| > 0$ and, if $n \geq 2$, $B(\mu)$ is an $(n-2)\times(n-2)$-matrix such that

$$\|e^{Bt}\| = o(e^{-\lambda t}) \quad (t \to +\infty). \tag{5}$$

Note also that condition (C) implies

$$\gamma - \lambda < 0. \tag{6}$$

By (4), the periodic orbit $L(\mu)$ is given by $x = 0$, $y = 0$, $z = 0$, its local stable manifold is given by $z = 0$, and the leading direction in the stable manifold is given by $y = 0$; the local unstable manifold is given by $\{x = 0, y = 0\}$.

Recall that the 4π-periodic transformation we used to bring system (3) to the autonomous form (4) is, in fact, 2π-periodic or 2π-antiperiodic. Namely, the points (θ, x, z, y) and $(\theta + 2\pi, \sigma(x, z, y))$ are equal (they represent the same point in the solid torus U), where σ is an involution which changes signs of some of the coordinates $x, z, y_1, \ldots, y_{n-2}$. More precisely, σ changes the orientation of each of the directions which correspond to the real negative multipliers ρ. In particular, if all the multipliers ρ are positive, then σ is the identity, i.e. our coordinates are 2π-periodic in this case.

(E) Consider two cross-sections $S_0 : \{x = d, \|y\| \leq \varepsilon_1, |z| \leq \varepsilon_1\}$ and $S_1 : \{z = d, \|y\| \leq \varepsilon_2, |x| \leq \varepsilon_2\}$ for some small positive d and $\varepsilon_{1,2}$. Denote the coordinates on S_0 as (y_0, z_0, θ_0) and the coordinates on S_1 as (x_1, y_1, θ_1). The set S_0 is divided by the stable manifold W^s into two regions, $S_0^+ : \{z_0 > 0\}$ and $S_0^- : \{z_0 < 0\}$. Since $W_0^+ \subset W_0^s$ by assumption 2, it follows that the orbits starting at S_1 define a smooth map $T_1 : S_1 \to S_0$ (see Fig. 3) for all small μ:

$$\begin{aligned}&z_0 = f(x_1, y_1, \theta_1, \mu)\\&y_0 = g(x_1, y_1, \theta_1, \mu)\\&\theta_0 = m\theta_1 + h(\theta_1, \mu) + \tilde{h}(x_1, y_1, \theta_1, \mu),\end{aligned} \tag{7}$$

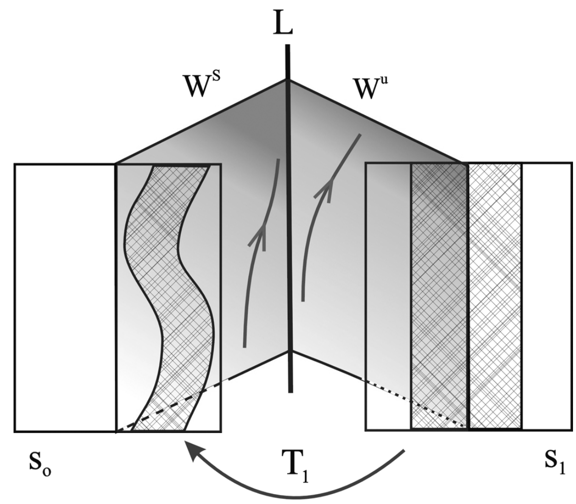

Fig. 3. Poincaré map T_1 takes a cross-section S_1 transverse to the unstable manifold W^u to a cross-section S_0 transverse to the stable manifold W^s.

where $f, g, h, \tilde{h}$ are smooth functions 4π-periodic in θ_1, and the function $\tilde{h}$ vanishes at $(x_1 = 0, y_1 = 0)$. Condition $W_0^+ \subset W_0^s$ reads as

$$f(0, 0, \theta_1, 0) \equiv 0.$$

We assume that

$$f(0, 0, \theta_1, \mu) = \mu\alpha(\theta_1, \mu), \tag{8}$$

where

$$\alpha(\theta_1, \mu) > 0 \tag{9}$$

for all θ_1, i.e. *all the homoclinics are split simultaneously and in the same direction*, and the intersection $W_\mu^+ \cap S_0$ moves inside S_0^+ with a nonzero velocity as μ grows across zero.

The coefficient m in the last equation of (7) is an integer. In order to see this, recall that two points (θ, x, z, y) and $(\hat{\theta}, \hat{x}, \hat{z}, \hat{y})$ in U are the same if and only if $\hat{\theta} = \theta + 2\pi k, (\hat{x}, \hat{z}, \hat{y}) = \sigma^k(x, z, y)$ for an integer k. Thus, if we increase θ_1 to 4π in the right-hand side of (7), then the corresponding value of θ_0 in the left-hand side may change only to an integer multiple of 2π, i.e. m must be an integer or a half-integer. Let us show that the half-integer m are forbidden by our assumption (9). Indeed, if the multiplier ρ_n is positive, then the involution σ keeps the corresponding variable z constant. Thus, $(z = d, \theta = \theta_1, x = 0, y = 0)$ and $(z = d, \theta = \theta_1 + 2\pi, x = 0, y = 0)$ correspond, in this case, to the same point on $W_\mu^+ \cap S_1$, hence their image by (7) must give the same point on S_0, i.e. the corresponding values of θ_0 must differ on an integer multiple of 2π, which means that m must be an integer. If $\rho_n < 0$, then σ changes the sign of z, i.e. if two values of θ_0 which correspond to the same point on S_0 differ on $2\pi k$, the corresponding values of z differ to a factor of $(-1)^k$. Now, since the increase of θ_1 to 4π leads to the increase of θ_0 to $4\pi m$ in (7), we find that $f(0,0,4\pi,\mu) = (-1)^{2m} f(0,0,0,\mu)$ in the case $\rho_n < 0$. This implies that if m is a half-integer, then $f(0,0,\theta)$ must have zeros at any μ and (9) cannot be satisfied.

The number m determines the shape of $W^+ \cap S_0$. Namely, the equation of the curve $W_0^+ \cap S_0$ is

$$\theta_0 = m\theta_1 + h_1(\theta_1, 0), \quad y_0 = g(0,0,\theta_1,0), \quad z_0 = 0,$$

so $|m|$ defines the homotopic type of this curve in $S_0 \cap W_0^s$, and the sign of m is responsible for the orientation. In the case $n = 2$, i.e. when the system is defined in R^3, the only possible case is $m = 1$. At $n = 3$ (the system in R^4) the curve $W_0^+ \cap S_0$ lies in the two-dimensional intersection of W^s with S_0. This is either an annulus (if $\rho_1 > 0$), or a Möbius strip (if $\rho_1 < 0$). Since the smooth curve $W_0^+ \cap S_0$ cannot have self-intersections, it follows that the only possible cases are $m = 0, \pm 1$ when $W^s \cap S_0$ is a two-dimensional annulus and $m = 0, \pm 1, \pm 2$ when $W_0^+ \cap S_0$ is a Möbius strip. At large n (the system in R^5 and higher) all integer values of m are possible.

Now we can formulate the main results of the paper.

Theorem. *Let conditions* (A)–(E) *hold. Consider a sufficiently small neighborhood V of the homoclinic surface $\Gamma = W_0^+ \cap L$.*

(1) *If $m = 0$ and, for all θ,*

$$\left| h'(\theta, 0) - \frac{\alpha'(\theta, 0)}{\gamma\alpha(\theta, 0)} \right| < 1, \qquad (10)$$

then a single stable periodic orbit $\mathcal{L}_\mu$ is born as Γ splits. The orbit $\mathcal{L}_\mu$ exists at all small $\mu > 0$; its period and length tend to infinity as $\mu \to +0$. All orbits which stay in V for all positive times and which do not lie in the stable manifold of the saddle orbit L_μ tend to $\mathcal{L}_\mu$.

(2) *If $|m| = 1$ and, for all θ,*

$$1 + m\left[h'(\theta, 0) - \frac{\alpha'(\theta, 0)}{\gamma\alpha(\theta, 0)} \right] > 0, \qquad (11)$$

then a stable two-dimensional invariant torus (at $m = 1$) or a Klein bottle (at $m = -1$) is born as Γ splits. It exists at all small $\mu > 0$ and attracts all the orbits which stay in V and which do not lie in the stable manifold of L_μ.

(3) *If $|m| \geq 2$ and, for all θ,*

$$\left| m + h'(\theta, 0) - \frac{\alpha'(\theta, 0)}{\gamma\alpha(\theta, 0)} \right| > 1, \qquad (12)$$

then, for all small $\mu > 0$, the system has a hyperbolic attractor (a Smale–Williams solenoid) which is an ω-limit set for all orbits which stay in V and which do not lie in the stable manifold of L_μ. The flow on the attractor is topologically conjugate to suspension over the inverse spectrum limit of a degree-m expanding map of a circle. At $\mu = 0$, the attractor degenerates into the homoclinic surface Γ.

Proof. Solution of (4) with the initial conditions $(x_0 = d, y_0, z_0, \theta_0) \in S_0$ gives

$$x(t) = e^{-\lambda t} d, \quad y(t) = e^{Bt} y_0,$$

$$z(t) = e^{\gamma t} z_0, \quad \theta(t) = \theta_0 + t.$$

The flight time to S_1 is found from the condition

$$d = e^{\gamma t} z_0,$$

which gives $t = -\frac{1}{\gamma} \ln \frac{z_0}{d}$. Thus the orbits in U define the map $T_0 : S_0^+ \to S_1$:

$$x_1 = d^{1-\nu} z_0^\nu, \quad y_1 = Q(z_0) y_0,$$

$$\theta_1 = \theta_0 - \frac{1}{\gamma} \ln \frac{z_0}{d}$$

where $\nu = \lambda/\gamma > 1$ and $\|Q(z_0)\| = o(z_0^\nu)$ [see (5) and (6)]. By (7), we may write the map $T = T_0 T_1$ on S_1 as follows (we drop the index "1"):

$$\bar{x} = d^{1-\nu} (\mu\alpha(\theta, \mu) + O(x, y))^\nu,$$

$$\bar{y} = Q(\mu\alpha + O(x, y)) g(x, y, \theta, \mu),$$

$$\bar{\theta} = m\theta + h(\theta, \mu)$$

$$\quad - \frac{1}{\gamma} \ln\Big(\frac{\mu}{d} \alpha(\theta, \mu) + O(x, y) \Big) + O(x, y).$$

For every orbit which stays in V, its consecutive intersections with the cross-section S_1 constitute an orbit of the diffeomorphism T. Since $\nu > 1$, the map T is contracting in x and y, and it is easy to see that all the orbits eventually enter a neighborhood of $(x, y) = 0$ of size $O(\mu^\nu)$. We therefore rescale the coordinates x and y as follows:

$$x = d^{1-\nu}\mu^\nu X, \quad y = \mu^\nu Y.$$

The map T takes the form

$$\begin{aligned}
\overline{X} &= \alpha(\theta, 0)^\nu + o(1),\\
\overline{Y} &= o(1),\\
\bar{\theta} &= \omega(\mu) + m\theta + h(\theta, 0)\\
&\quad -\frac{1}{\gamma}\ln\alpha(\theta, 0) + o(1),
\end{aligned} \tag{13}$$

where $o(1)$ stands for terms which tend to zero as $\mu \to +0$, along with their first derivatives, and $\omega(\mu) = \frac{1}{\gamma}\ln(\mu/d) \to \infty$ as $\mu \to +0$. Recall that $\alpha > 0$ for all θ and that α and h are periodic in θ.

It is immediately seen from (13) that all orbits eventually enter an invariant solid torus $\{|x - \alpha(\theta, 0)^\nu| < K_\mu, \|y\| < K_\mu\}$ for appropriately chosen K_μ, $K_\mu \to 0$ as $\mu \to +0$ (see Fig. 4). Thus, there is an attractor in V for all small positive μ, and it merges into Γ as $\mu \to +0$. Our theorem claims that the structure of the attractor depends on the value of m, so we now consider different cases separately.

If $m = 0$ and (10) holds, then map (13) is, obviously, contracting at small μ, hence it has a single stable fixed point. This fixed point corresponds to the sought periodic orbit A_μ. Its period tends to infinity as $\mu \to +0$: the orbit intersects both the cross-sections S_0 and S_1, and the flight time from S_0 to S_1 is of order $\frac{1}{\gamma}|\ln\mu|$. The length of the orbit also tends to infinity, since the phase velocity never vanishes in V.

In the case $m = \pm 1$ we prove the theorem by referring to the "annulus principle" of [Afraimovich & Shilnikov, 1978]. Namely, consider a map

$$\bar{r} = p(r, \theta), \quad \bar{\theta} = q(r, \theta)$$

of a solid torus into itself (here θ is the angular variable and r is the vector of normal variables). Let the map $r \mapsto p(r, \theta)$ be a contraction for every fixed θ, i.e.

$$\left\|\frac{\partial p}{\partial r}\right\|_\circ < 1$$

(where by $\|\cdot\|_\circ$ we denote the supremum of the norm over the solid torus under consideration) and let the map $\theta \mapsto q(r, \theta)$ be a diffeomorphism of a circle for every fixed r. Then it is well-known [Afraimovich & Shilnikov, 1978; Shilnikov *et al.*, 2001] that if

$$1 - \left\|\left(\frac{\partial q}{\partial \theta}\right)^{-1}\right\|_\circ \cdot \left\|\frac{\partial p}{\partial r}\right\|_\circ > 2\sqrt{\left\|\left(\frac{\partial q}{\partial \theta}\right)^{-1}\right\|_\circ \cdot \left\|\frac{\partial q}{\partial r}\right\|_\circ \left\|\frac{\partial p}{\partial \theta}\left(\frac{\partial q}{\partial \theta}\right)^{-1}\right\|_\circ},$$

then the map has a stable, smooth, closed invariant curve $r = r^*(\theta)$ which attracts all orbits from the solid torus. These conditions are clearly satisfied by map (13) at $|m| = 1$ if (11) is true (here $r = (X, Y)$, $p = (\alpha(\theta, 0)^\nu + o(1), o(1))$, $q = \omega(\mu) + m\theta + h(\theta, 0) - \frac{1}{\gamma}\ln\alpha(\theta, 0) + o(1))$. Thus, the map T has a closed invariant curve in this case. The restriction of T to the invariant curve preserves orientation if $m = 1$, while at $m = -1$, it is orientation-reversing. Therefore, this invariant

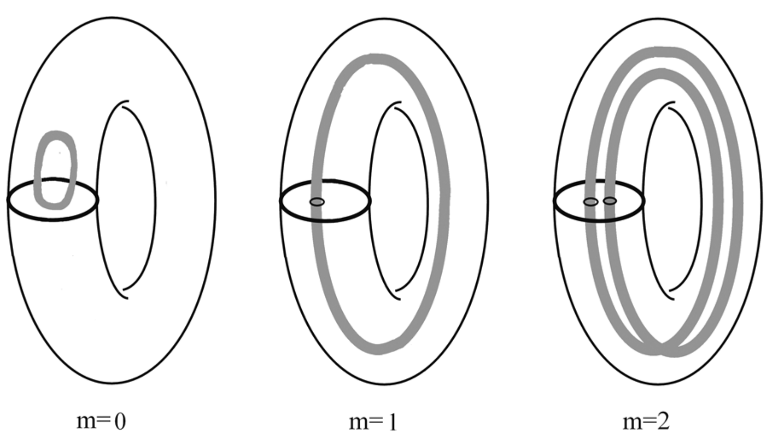

Fig. 4. Case $m = 0$: the image of the solid torus is contractible to a point; case $m = 1$: contraction transverse to the longitude; case $m = 2$: the solid-torus is squeezed, doubly stretched and twisted within the original and so on, producing the solenoid in the limit.

curve on the cross-section corresponds to an invariant torus of the flow at $m = 1$ or to a Klein bottle at $m = -1$.

It remains to prove the theorem for the case $|m| \geq 2$. The proof is based on the following result.

Lemma. *Consider a diffeomorphism* $T : (r, \theta) \mapsto (\bar{r}, \bar{\theta})$ *of a solid torus, where*

$$\bar{r} = p(r, \theta), \quad \bar{\theta} = m\theta + s(r, \theta) = q(r, \theta), \tag{14}$$

where s *and* p *are periodic functions of* θ. *Let* $|m| \geq 2$, *and*

$$\left\| \frac{\partial p}{\partial r} \right\|_\circ < 1, \tag{15}$$

$$\left(1 - \left\| \frac{\partial p}{\partial r} \right\|_\circ\right) \left(1 - \left\| \left(\frac{\partial q}{\partial \theta}\right)^{-1} \right\|_\circ\right) > \left\| \frac{\partial p}{\partial \theta} \right\|_\circ \left\| \left(\frac{\partial q}{\partial \theta}\right)^{-1} \frac{\partial q}{\partial r} \right\|_\circ. \tag{16}$$

Then the map has a uniformly-hyperbolic attractor, a Smale–Williams solenoid, on which it is topologically conjugate to the inverse spectrum limit of $\bar{\theta} = m\theta$, *a degree-*m *expanding map of the circle.*

Proof. It follows from (15) and (16) that $\|(\frac{\partial q}{\partial \theta})^{-1}\|$ is uniformly bounded. Therefore, θ is a uniquely defined smooth function of $(\bar{\theta}, r)$, so we may rewrite (14) in the "cross-form"

$$\bar{r} = p^\times(r, \bar{\theta}), \quad \theta = q^\times(r, \bar{\theta}), \tag{17}$$

where $p^\times$ and $q^\times$ are smooth functions. It is easy to see that conditions (15) and (16) imply

$$\left\| \frac{\partial p^\times}{\partial r} \right\|_\circ < 1, \quad \left\| \frac{\partial q^\times}{\partial \bar{\theta}} \right\|_\circ < 1 \tag{18}$$

$$\left(1 - \left\| \frac{\partial p^\times}{\partial r} \right\|_\circ\right) \left(1 - \left\| \frac{\partial q^\times}{\partial \theta} \right\|_\circ\right) \geq \left\| \frac{\partial p^\times}{\partial \bar{\theta}} \right\|_\circ \left\| \frac{\partial q^\times}{\partial r} \right\|_\circ. \tag{19}$$

These inequalities imply the uniform hyperbolicity of the map T (note that (16) coincides with the hyperbolicity condition for the Poincaré map for the Lorenz attractor from [Afraimovich *et al.*, 1983]). Indeed, it is enough to show that there exists $L > 0$ such that the derivative T' of T takes every cone $\|\Delta r\| \leq L \|\Delta\theta\|$ inside $\|\Delta\bar{r}\| \leq L\|\Delta\bar{\theta}\|$ and is uniformly expanding in θ in this cone, and that the inverse of T' takes every cone $\|\Delta\bar{\theta}\| \leq L^{-1}\|\Delta\bar{r}\|$ inside $\|\Delta\theta\| \leq L^{-1}\|\Delta r\|$ and is uniformly expanding in r in this cone. Let us check these properties. When $\|\Delta r\| \leq L\|\Delta\theta\|$, we find from (17) that

$$\|\Delta\theta\| \leq \frac{\left\| \frac{\partial q^\times}{\partial \bar{\theta}} \right\|_\circ}{1 - L \left\| \frac{\partial q^\times}{\partial r} \right\|_\circ} \|\Delta\bar{\theta}\| \tag{20}$$

and

$$\|\Delta\bar{r}\| \leq \left\{ \frac{L \left\| \frac{\partial p^\times}{\partial r} \right\|_\circ \left\| \frac{\partial q^\times}{\partial \bar{\theta}} \right\|_\circ}{1 - L \left\| \frac{\partial q^\times}{\partial r} \right\|_\circ} + \left\| \frac{\partial p^\times}{\partial \bar{\theta}} \right\|_\circ \right\} \|\Delta\bar{\theta}\|. \tag{21}$$

Similarly, if $\|\Delta\bar{\theta}\| \leq L^{-1}\|\Delta\bar{r}\|$, we find from (17) that

$$\|\Delta\bar{r}\| \leq \frac{\left\| \frac{\partial p^\times}{\partial r} \right\|_\circ}{1 - L^{-1} \left\| \frac{\partial p^\times}{\partial \bar{\theta}} \right\|_\circ} \|\Delta r\| \tag{22}$$

and

$$\|\Delta\theta\| \leq \left\{ \frac{L^{-1} \left\| \frac{\partial q^\times}{\partial \bar{\theta}} \right\|_\circ \left\| \frac{\partial p^\times}{\partial r} \right\|_\circ}{1 - L^{-1} \left\| \frac{\partial p^\times}{\partial \bar{\theta}} \right\|_\circ} + \left\| \frac{\partial q^\times}{\partial r} \right\|_\circ \right\} \|\Delta r\|. \tag{23}$$

Thus, we will prove hyperbolicity if we show that there exists L such that

$$\left\| \frac{\partial q^\times}{\partial \bar{\theta}} \right\|_\circ < 1 - L \left\| \frac{\partial q^\times}{\partial r} \right\|_\circ$$

and

$$\left\| \frac{\partial p^\times}{\partial r} \right\|_\circ < 1 - L^{-1} \left\| \frac{\partial p^\times}{\partial \bar{\theta}} \right\|_\circ.$$

These conditions are solved by any L such that

$$\frac{\left\| \frac{\partial p^\times}{\partial \bar{\theta}} \right\|_\circ}{1 - \left\| \frac{\partial p^\times}{\partial r} \right\|_\circ} < L < \frac{1 - \left\| \frac{\partial q^\times}{\partial \bar{\theta}} \right\|_\circ}{\left\| \frac{\partial q^\times}{\partial r} \right\|_\circ}.$$

It remains to note that such L exist indeed when (18) and (19) are satisfied. ■

We have proved that the attractor A of the map T is uniformly hyperbolic. Such attractors are structurally stable, so $T|_A$ is topologically conjugate to the restriction to the attractor of any diffeomorphism which can be obtained by a continuous deformation of the map T without violation of conditions (15) and (16). An obvious example of such a diffeomorphism is given by the map

$$\bar{r} = p(\delta r, \theta), \quad \bar{\theta} = q(\delta r, \theta) \tag{24}$$

for any $0 < \delta \leq 1$. Fix small $\delta > 0$ and consider a family of maps

$$\bar{r} = p(\delta r, \theta), \quad \bar{\theta} = q(\varepsilon r, \theta),$$

where ε runs from δ to zero. When δ is sufficiently small, every map in this family is a diffeomorphism (otherwise we would get that the curve $\{\bar{r} = p(0, \theta), \bar{\theta} = q(0, \theta)\}$ would have points of self-intersection, which is impossible since this curve is the image of the circle $r = 0$ by the diffeomorphism T), and each satisfies inequalities (15) and (16). This family is a continuous deformation of map (24) to the map

$$\bar{r} = p(\delta r, \theta), \quad \bar{\theta} = q(0, \theta) = m\theta + s(0, \theta). \tag{25}$$

Thus, we find that $T|_A$ is topologically conjugate to the restriction of diffeomorphism (25) to its attractor. It remains to note that map (25) is a skew-product map of the solid torus, which contracts along the fibers $\theta =$ const and, in the base, it is an expanding degree-m map of a circle. By definition, the attractor of such map is the sought Smale–Williams solenoid [Smale, 1967; Williams, 1974]. This completes the proof of the lemma.

Now, in order to finish the proof of the theorem, just note that map (13) satisfies the conditions of the lemma when (12) is fulfilled.

Acknowledgment

This work was in part supported by NSF DMS-1009591 and RFFI 11-01-00001 (to A. Shilnikov), and the Royal Society Grant "Homoclinic bifurcations" (to D. Turaev).

References

Abraham, R. H. [1985] *Catastrophes, Intermittency, and Noise in Chaos, Fractals, and Dynamics*, Lecture Notes in Pure and Applied Mathematics, Vol. 98, pp. 3–22.

Afraimovich, V. S. & Shilnikov, L. P. [1974a] "On small periodic perturbations of autonomous systems," *Sov. Math. Dokl.* **15**, 206–211.

Afraimovich, V. S. & Shilnikov, L. P. [1974b] "On some global bifurcations connected with the disappearance of a fixed point of saddle-node type," *Sov. Math. Dokl.* **15**, 1761–1765.

Afraimovich, V. S. & Shilnikov, L. P. [1978] "The annulus principle and problems of interaction between two self-oscillating systems," *J. Appl. Math. Mech.* **41**, 632–641.

Afraimovich, V. S. & Shilnikov, L. P. [1982] "On bifurcation of codimension-one, leading to the appearance of a countable set of tori," *Sov. Math. Dokl.* **25**, 101–105.

Afraimovich, V. S., Bykov, V. V. & Shilnikov, L. P. [1983] "On attracting structurally unstable limit sets of Lorenz attractor type," *Trans. Moscow Math. Soc.* **44**, 153–216.

Afraimovich, V. S. & Shilnikov, L. P. [1991] *Invariant Two-Dimensional Tori, Their Breakdown and Stochasticity*, American Mathematical Society Translations, Vol. 149, pp. 201–211.

Andronov, A. A. & Leontovich, E. A. [1937] "Some cases of dependence of limit cycles on a parameter," *Uchenye zapiski Gorkovskogo Universiteta* (*Research Notes of Gorky University*) **6**, 3–24.

Andronov, A. A., Leontovich, E. A., Gordon, I. E. & Maier, A. G. [1971] *The Theory of Bifurcations of Dynamical Systems on a Plane* (Wiley, NY).

Belykh, I. V. & Shilnikov, A. L. [2008] "When weak inhibition synchronizes strongly desynchronizing networks of bursting neurons," *Phys. Rev. Lett.* **101**, 078102.

Belykh, I., Jalil, S. & Shilnikov, A. L. [2010] "Burst-duration mechanism of in-phase bursting in inhibitory networks," *Regul. Chaot. Dyn.* **15**, 148–160.

Fuller, F. [1967] "An index of fixed point type for periodic orbits," *Amer. J. Math.* **89**, 133–148.

Gavrilov, N. K. & Shilnikov, A. L. [2000] *Example of a Blue Sky Catastrophe*, American Mathematical Society Translations, Series II, Vol. 200, pp. 99–105.

Glyzin, S. D., Kolesov, A. Yu. & Rozov, N. Kh. [2008] "Blue sky catastrophe in relaxation systems with one fast and two slow variables," *Diff. Eqs.* **44**, 161–175.

Ilyashenko, Y. & Li, W. [1999] *Nonlocal Bifurcations*, Math. Surveys and Monographs, Vol. 66 (AMS).

Li, W. & Zhang, Z. F. [1991] "The blue sky catastrophe on closed surfaces," *Adv. Series Dyn. Syst.* **9**, 316–332.

Medvedev, V. S. [1982] "On a new type of bifurcations on manifolds," *Math. USSR Sb.* **41**, 403–407.

Newhouse, S., Palis, J. & Takens, F. [1983] "Bifurcations and stability of families of diffeomorphisms," *Publ. Math. IHES* **57**, 5–71.

Palis, J. & Pugh, Ch. [1975] "Fifty problems in dynamical systems," *Dynamical Systems*, Warwick, 1974, Springer Lecture Notes, Vol. 468.

Shilnikov, L. P. [1963] "Some cases of generation of periodic motion from singular trajectories," *Math. USSR Sb.* **61**, 443–466.

Shilnikov, L. P. [1966] "On the generation of a periodic motion from a trajectory which leaves and re-enters a saddle-saddle state of equilibrium," *Sov. Math. Dokl.* **7**, 1155–1158.

Shilnikov, L. P. [1968] "On the generation of a periodic motion from trajectories doubly asymptotic to an equilibrium state of saddle type," *Math. USSR Sb.* **6**, 427–438.

Shilnikov, L. P. [1991] "The theory of bifurcations and turbulence," *Selecta Math. Sovietica* **10**, 43–53.

Shilnikov, L. P. & Turaev, D. V. [1997] "On simple bifurcations leading to hyperbolic attractors," *Comput. Math. Appl.* **34**, 441–457.

Shilnikov, L. P. & Turaev, D. V. [2000] "A new simple bifurcation of a periodic orbit of blue sky catastrophe type," in *Methods of Qualitative Theory of Differential Equations and Related Topics*, American Mathematical Society Translations, Series II, Vol. 200, pp. 165–188.

Shilnikov, L. P., Shilnikov, A. L., Turaev, D. V. & Chua, L. O. [2001] *Methods of Qualitative Theory in Nonlinear Dynamics. Part II* (World Scientific, Singapore).

Shilnikov, A. L. & Cymbalyuk, G. [2004] "Homoclinic saddle-node orbit bifurcations en route between tonic spiking and bursting in neuron models," *Regul. Chaot. Dyn.* **3**, 281–297.

Shilnikov, A. L., Shilnikov, L. P. & Turaev, D. V. [2004] "On some mathematical aspects of classical synchronization theory. Tutorial," *Int. J. Bifurcation and Chaos* **14**, 2143–2160.

Shilnikov, A. L., Shilnikov, L. P. & Turaev, D. V. [2005] "Blue sky catastrophe in singularly perturbed systems," *Moscow Math. J.* **5**, 205–218.

Shilnikov, A. L. & Cymbalyuk, G. [2005] "Transition between tonic-spiking and bursting in a neuron model via the blue-sky catastrophe," *Phys. Rev. Lett.* **94**, 048101.

Shilnikov, A. L. & Turaev, D. V. [2006] "Blue sky catastrophe," *Scholarpedia* **2**, 1889.

Shilnikov, A. L. & Kolomiets, M. L. [2008] "Methods of the qualitative theory for the Hindmarsh–Rose model: A case study," *Int. J. Bifurcation and Chaos* **18**, 1–27.

Smale, S. [1967] "Differentiable dynamical systems," *Bull. Amer. Math. Soc.* **73**, 747–817.

Turaev, D. V. & Shilnikov, L. P. [1986] "Bifurcations of quasiattractors torus-chaos," in *Mathematical Mechanisms of Turbulence* (*Modern Nonlinear Dynamics in Application to Turbulence Simulation*), Kiev, pp. 113–121.

Turaev, D. V. & Shilnikov, L. P. [1995] "On blue sky catastrophes," *Dokl. Math.* **51**, 404–407.

Williams, R. F. [1974] "Expanding attractors," *Publ. Math. IHES* **43**, 169–203.

Wojcik, J., Clewley, R. & Shilnikov, A. L. [2011] "Order parameter for bursting polyrhythms in multifunctional central pattern generators," *Phys. Rev. E* **83**, 056209.

Wojcik, J., Schwabedal, J., Clewley, R. & Shilnikov, A. L. [2014] "Key bifurcations of bursting polyrhythms in 3-cell central pattern generators," *PLoS ONE* **9**, e92918.