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Scientific Heritage of L. P. Shilnikov. Part II. Homoclinic Chaos

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Abstract—We review the works initiated and developed by L. P. Shilnikov on homoclinic chaos, highlighting his fundamental contributions to Poincaré homoclinics, periodic orbits, and invariant tori. Additionally, we discuss his related findings in non-autonomous and infinite-dimensional systems. This survey continues our earlier review [1], where we examined Shilnikov's groundbreaking results on bifurcations of homoclinic orbits — his extension of the classical work by A. A. Andronov and E. A. Leontovich from planar to multidimensional autonomous systems, as well as his pioneering discoveries on saddle-focus loops and spiral chaos.

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In 2024 we celebrated the 90th anniversary of the outstanding Russian mathematician Leonid Pavlovich Shilnikov (1934–2011). The anniversary became a motivation for us to continue the review of scientific results of Shilnikov, which we began 10 years ago [1]. Fundamental contributions of Shilnikov include the creation of the theory of global bifurcations of multidimensional dynamical systems, the discovery of spiral chaos, the theory of the Lorenz attractor and its generalizations, the discovery of the breakdown of an invariant torus as the main mechanism of the transition from quasiperiodicity to chaos, see the review [2]. The core result of his scientific work was the complete description of the structure of a neighborhood of a transverse homoclinic orbit.

1. HOMOCLINIC ORBITS AND A POINCARÉ – BIRKHOFF PROBLEM

One of the most fundamental results of the theory of dynamical chaos is the Smale-Shilnikov theorem about the dynamics near a transverse Poincaré homoclinic orbit. This is an orbit that belongs to the intersection of the stable and unstable invariant manifolds W^s and W^u of a saddle periodic orbit L. A homoclinic orbit Γ is called transverse when the manifolds $W^s(L)$ and $W^u(L)$

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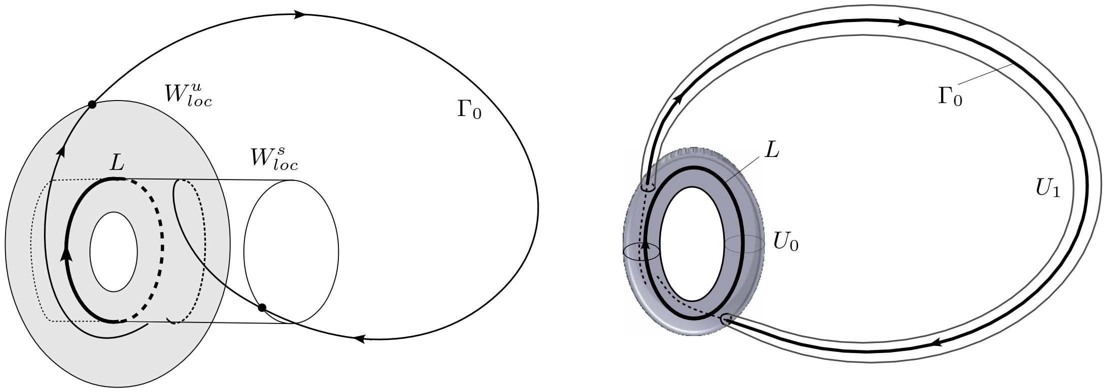


(a) (b)

Fig. 1. (a) A homoclinic curve Γ to a saddle periodic orbit L whose stable $W^s(L)$ and unstable $W^u(L)$ invariant manifolds intersect transversely along Γ. (b) A neighborhood $U = U(L \cup \Gamma_0)$ of the homoclinic curve Γ: here $U = U_0 \cup U_1$ is a solid torus U_0 around L with a handle U_1 around Γ.

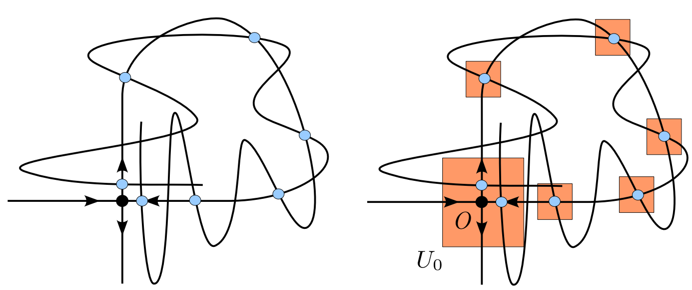
are transverse to each other at the points of Γ. In the case of continuous time, Γ_0 is also called a *Poincaré homoclinic curve*, see Fig. 1.

If an $n+1$-dimensional smooth (at least C^1) system of differential equations has a saddle periodic orbit L with multipliers ρ_i, $i = 1, \ldots, n$, such that $|\rho_i| < 1, i = 1, \ldots, k$, and $|\rho_i| > 1, i = k+1, \ldots, n$, then the stable manifold $W^s(L)$ is $(k+1)$-dimensional and the unstable manifold $W^u(L)$ is $(n-k+1)$-dimensional. The manifold W^s consists of orbits that tend to L as $t \to +\infty$, and W^u consists of orbits that tend to L as $t \to -\infty$. Therefore, the homoclinic curve Γ is an orbit (other than L) which tends to L both in forward and backward time.

If the flow admits a global cross-section S, then the study of solutions near the homoclinic curve is reduced to that of the corresponding Poincaré map $T : S \mapsto S$ by the orbits of the flow. In particular, the global cross-section exists for nonautonomous systems $\dot{x} = X(x,t),\ \ x \in R^n$, when the right-hand side depends periodically on time. In this case, one takes $S : \{t = 0\}$, and the Poincaré map T is the map over the period of $X(x,t)$.

In general, a homoclinic orbit in the discrete-time case is defined as follows. Let an n-dimensional diffeomorphism T have a saddle fixed (or periodic) point O with multipliers $\rho_1, \ldots, \ldots, \rho_n$, such that $|\rho_i| < 1, i = 1, \ldots, k$, and $|\rho_i| > 1, i = k+1, \ldots, n$. Then O has smooth stable and unstable invariant manifolds $W^s(O)$ and $W^u(O)$ (of dimensions k and $n-k$, respectively). In this case, if $M \in W^s \cap W^u$ is a homoclinic point, then the corresponding homoclinic orbit is defined as $\Gamma = \{T^n M\}_{n=-\infty}^{+\infty}$, see Fig. 2.

In the mid-60s L. P. Shilnikov set up and solved the problem of a complete description of the set N of all orbits that stay near a transverse homoclinic orbit Γ. He called it the *Poincaré–Birkhoff problem.* The solution is given by



(a) (b)

Fig. 2. (a) A saddle fixed point O with a transverse homoclinic orbit Γ for a 2D diffeomorphism, and (b) the neighborhood $U = U(O \cup \Gamma)$ defined as the union of a small disk U_0 containing O and a number of small disks of those points of Γ that do not belong to U_0.

Theorem 1 (Shilnikov Theorem [3]). *The set N of all orbits that lie entirely in a sufficiently small neighborhood U of the closure of the transverse homoclinic orbit Γ is a uniformly hyperbolic set, which is in one-to-one correspondence with the set of orbits of the Bernoulli shift $\mathcal{B}_2$ of two symbols.*[1)]

Let us explain this statement. Consider first the continuous-time case. The closure of Γ, by definition, is $L \cup \Gamma$. The neighborhood U of $L \cup \Gamma$ is a solid torus U_0 around L with a handle U_1 around $\Gamma \backslash U_0$, Fig. 1b. Any trajectory that stays within U can be symbolically represented by an infinite sequence of 0's and 1's constructed by the following rule: the symbol 0 corresponds to each round the orbit makes in U_0, while a passage within the handle U_1 is coded by the symbol 1. In particular, the periodic orbit L is coded by an infinite sequence of zeros

$$(\dots, 0, 0, 0, \dots),$$

whereas the symbolic representation of the homoclinic orbit Γ is given by the sequence

$$(\dots, 0, 1, 0, \dots).$$

Note that we consider a small neighborhood U, which means, in particular, that the handle U_1 is thin enough . This implies that an orbit entering U_0 from U_1 must make many rounds in U_0 before entering U_1 again. In other words, the symbol 1 in the coding sequence is necessarily followed by a string of zeros of the length at least $\bar{n}$, where the integer $\bar{n}$ depends on the choice of the neighborhood U (the smaller the neighborhood, the larger $\bar{n}$).

The essence of the Shilnikov Theorem is that
every symbolic sequence satisfying this condition corresponds to some orbit lying in U, and this orbit is unique.

By construction, the restriction of the system to the invariant set N is topologically equivalent to the suspension[2)] over the topological Bernoulli shift of two symbols, Fig. 3a. In the discrete-time case, for a diffeomorphism T with a transverse homoclinic, the Shilnikov Theorem gives that
the dynamical system $T|_N$ is topologically conjugate with the topological Markov chain defined by the graph shown in Fig. 3b.

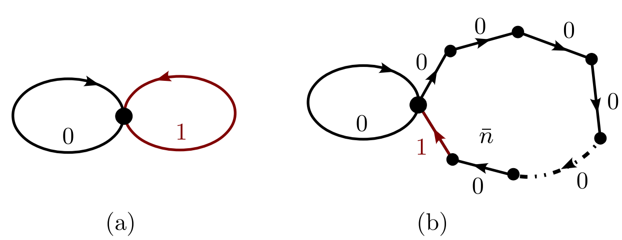


Fig. 3. (a) The simplest graph of the topological Bernoulli shift of two symbols. (b) Graph of the Shilnikov TMC, equivalent to the "Birkhoff – Morse system" of [3].

Obviously, the equivalence to the Bernoulli shift means the chaotic dynamics in the very sense of the words. In particular, we have here a sensitive dependence of initial conditions, an infinite number of periodic orbits, uncountably many nontrivial recurrent orbits, and positive topological entropy. Shilnikov himself considered this result to be of fundamental importance and always emphasized that *the existence of a transverse homoclinic orbit is a universal criterion for chaos* (see, e. g., his review [5]).

1) The original formulation (Theorems 8.1 and 8.2 in [3]) employed infinitely many symbols, but it is more convenient here to formulate the result in this equivalent form.

2) The suspensions over topological Markov chains (TMC) were systematically studied in [7, 8], where conditions of topological equivalence between suspensions over different TMCs were given, in particular. Recall that a suspension over a discrete dynamical system with a compact phase space D is a flow for which D is a global cross-section, see, e. g., [9].

We must also add that this was an exemplary work that provided a universal method for establishing chaos: given an invariant set, take its small neighborhood, define the natural symbolic coding and prove the existence of hyperbolic sets with a sufficiently rich set of codes by applying the method of cross-maps (see Section 2). The method was systematically used by Shilnikov and his (our) school for the study of numerous global bifurcations and beyond.

1.1. A Bit of Homoclinic History

The full description obtained by Shilnikov for the chaotic dynamics near a transverse homoclinic orbit laid the foundation for the theory of chaotic behavior in multidimensional dynamical systems and completed a line of studies by Poincaré, Birkhoff, and Smale, among others.

Homoclinic orbits were discovered by H. Poincaré as far back as in 1890 in his famous memoir [4] on celestial mechanics[3)]. Poincaré showed that the presence of a homoclinic trajectory in a Hamiltonian system is an insurmountable obstacle to its integrability. He used this as one of the main arguments for the necessity of qualitative, i. e., geometrical or topological, methods for studying multidimensional dynamics. His three-volume book [11] was devoted to the development of such methods, mostly for nearly integrable Hamiltonian systems.

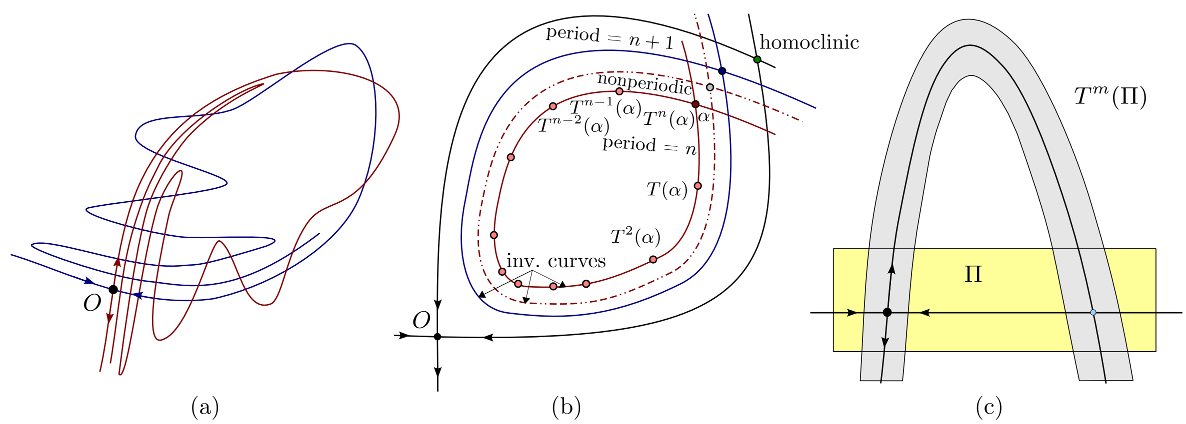


Fig. 4. Homoclinic orbits from Poincaré to Smale: (a) Poincaré homoclinic tangle; (b) schematics of the Birkhoff method: an infinite sequence of invariant curves whose self-intersection points are periodic points of the map; (c) a Smale horseshoe near a homoclinic crossing.

From the modern perspective, the discovery of Poincaré homoclinic orbits is universally accepted as the beginning of the dynamical chaos history. Formally speaking, Poincaré presented only one general result on homoclinic orbits, which can be rephrased as follows:

if a two-dimensional diffeomorphism has a transverse homoclinic orbit to a saddle periodic point, then there also exist infinitely many other homoclinic orbits to the same point.

Poincaré also noted the incredible complexity of what is now called a "homoclinic tangle", see Fig. 4a. He wrote [11]: "If we seek to represent the figure formed by the two curves [the stable and unstable manifolds] and their infinite number of intersections, each of which corresponds to a doubly asymptotic solution, these intersections form a kind of lattice, a fabric, an infinitely tight mesh; each of the two curves must never intersect itself, but must fold back on itself in a highly complex manner to intersect all the stitches of the mesh infinitely many times. We will be struck by the complexity of this figure, which I am not even trying to draw."

The existence of Poincaré homoclinic orbits for geodesic flows on surfaces of negative curvature was evident in the paper of Hadamard [14]. Hadamard's work is remarkable for many reasons; in

[3)] The reader can find many interesting comments about the history of this discovery in the Shilnikov's review [5], see also [12, 13].

particular, it is the first paper where the hyperbolic chaos was described. He also asked whether this type of dynamics can be relevant to celestial mechanics, however Hadamard's theory had been considered to be a part of Riemannian geometry for a long time. It was not until 65 years later that his theory was revisited by Anosov [15, 16] to give rise to the modern theory of hyperbolic chaos.

The study of dynamics near a transverse homoclinic was continued by G. Birkhoff who established in 1935 [17] that

an analytic area-preserving map has infinitely many periodic orbits in any neighborhood of a transverse homoclinic orbit.

Birkhoff's method is based on an interesting geometrical construction, see Fig. 4b. In a small neighborhood of a saddle fixed point O there exists a continuous family of analytic invariant curves of the form $xy = \text{const}$ in appropriate coordinates (where $W^s_{loc}(O)$ and $W^u_{loc}(O)$ have equations $y = 0$ and $x = 0$, respectively). Each such local invariant curve can be extended globally by forward and backward iterations of the map. Then, since the separatrices $W^s(O)$ and $W^u(O)$ intersect transversely, each of the globally extended invariant curves must have a self-intersection near the corresponding homoclinic point. The trajectory of the self-intersection point remains on the curve, because the curve is invariant. The closer the curve to the saddle O, the more iterations are necessary to return to a neighborhood of the homoclinic point. If follows that, for any sufficiently large n, there exists a curve in our family, such that its self-intersection point returns exactly to itself after n iterations, i. e., it is n-periodic.

Note that Birkhoff assumes that the map is brought, by some smooth coordinate transformation, to the following normal form near the point O:

$$\bar{x} = \lambda x\, U(xy), \quad \bar{y} = \lambda^{-1} y\, U^{-1}(xy).$$

Here $0 < \lambda < 1$ and U is a function of only one variable $z = xy$. This normal form gives that $\bar{x}\bar{y} = xy$, i. e., the curves $xy = const$ are locally invariant and the above geometrical argument goes through. However, Birkhoff did not prove that the formal series he constructed for the normalizing transformation is convergent. It was done later in the mid-50s by J. Moser [18]. Therefore, the proof of the Birkhoff result was, in fact, completed by J. Moser.

The breakthrough came later, in the works of Smale [20] and Shilnikov [3]. Smale proved that

if a sufficiently smooth diffeomorphism T has a hyperbolic saddle periodic point O (such that the period map is linear near O) and a homoclinic point x corresponding to a transverse intersection of $W^s(O)$ and $W^u(O)$, then there exists an integer $p > 0$ such that T^p has a compact hyperbolic invariant set Λ (Smale horseshoe) containing O and x, and $T^p|_\Lambda$ is topologically conjugate to the Bernoulli shift of 2 symbols.

The proof in [20] consists in the formal description of the following, intuitively appealing geometric figure (see Fig. 4c). Take a narrow rectangle Q around a piece of the stable manifold of O containing the point O and the homoclinic point x. It is clear that $T^p(Q)$ for a sufficiently large p has a horseshoe-like shape. Using the local linearization assumption Smale shows that the map $T^p|_Q$ is hyperbolic, hence, it is indeed the Smale horseshoe.

The work of Smale established the link between the Poincaré homoclinic tangle and the earlier examples of robustly chaotic behavior related to the Smale horseshoe [21]: geodesic flows on manifolds of negative curvature [14, 15], Cartwright-Littlewood chaos in the periodically driven van der Pol equation [22, 23], and linear hyperbolic automorphisms of a torus (for more on this, see in the historical essay [24]). It is also important that he invoked the methods of symbolic dynamics, that go back to Hadamard, to the study of chaos. The main drawback of [20] was the use of the linearization, which excluded, e. g., the case of Hamiltonian systems. Note also that Smale did not consider the question of the complete description of the neighborhood of the homoclinic orbit.

These issues were resolved by Shilnikov, see Theorem 1 above. To avoid the linearization assumption, he developed the cross-map method (we describe it in Section 2), which was later systematically used by him and his school for the rigorous study of chaotic dynamics emerging at various global bifurcations. The method was also used for the analysis of transverse homoclinics in the non-autonomous and infinite-dimensional cases [42**?** , 43], see Sections 4 and 4.

Immediately after [3], Shilnikov published a short paper [25] on this topic, in which he described the set of trajectories near a set of transverse homoclinic orbits to an invariant torus[4)]. This work was way ahead of its time. In modern terms, he showed that the set of all orbits that lie in a small neighborhood of of the strongly transverse homoclinic tube to a normally hyperbolic invariant torus is a product of the Bernoulli scheme to a torus. Dynamics on such set can be highly nontrivial, see [30–34]. This set also emerges at homoclinic bifurcations [28] when a periodic or qusiperiodic force is applied to a system with chaotic dynamics [35, 37]. The Shilnikov construction also found an application in the Arnold diffusion problem [29, 36]. We review the results of [25] in Section 3.

2. PROOF OF THE SHILNIKOV THEOREM

Let us describe the key steps of the proof of Theorem 1.

2.1. Symbolic Coding

We have already described how to code the orbits in a small neighborhood in the continuous-time case, see Fig. 1. Let us discuss the case of discrete time.

Let T be a diffeomorphism that has a saddle periodic trajectory $L = \{O, T(O), \ldots, T^{\ell-1}(O)\}$, where ℓ is the period of L. Let there exist a transverse homoclinic orbit Γ to L (Fig. 2a), and let $U = U(L \cup \Gamma_0)$ be its sufficiently small neighborhood (Fig. 2b). Note that U consists of a neighborhood U_0 of L (the union of ℓ disjoint balls) and a finite number of small balls around the points of $\Gamma \backslash U_0$. Let $\Lambda = \{P_i\}_{i=-\infty}^{+\infty}$ be an orbit that lies in U. It is coded as follows: we replace each point of Λ by “0” except for those points P_i which do not belong to U_0 but $T(P_i) \in U_0$ — these points are replaced by “1”.

Since Γ is doubly asymptotic to O, we can choose two points M^+ and M^- in a small neighborhood of the point O such that $M^+ \in W^s_{loc}(O)$, $M^- \in W^u_{loc}(O)$, and both M^+ and M^- belong to Γ, so $M^+ = T^p(M^-)$ for some $p > 0$. We can choose the neighborhood U_0 such that $M^\pm \in U_0$, $T(M^-) \notin U_0$ and $T^{-1}(M^+) \notin U_0$. Choose in U_0 sufficiently small neighborhoods Π^+ and Π^- of the points M^+ and M^-. Denote $T_1 = T^q|_{\Pi^-}$. We also denote $T_0 = T^\ell$ (restricted to a small neighborhood of O). The maps T_0 and T_1 are called local and global maps, respectively.

In the continuous-time case, one can also define the local and global maps: T_0 is the return map to a cross-section S to L by orbits in a small neighborhood of L, and T_1 is the map from a small neighborhood of $M^+ \in W^s_{loc}(L) \cap S$ to a small neighborhood of $M^- \in W^u_{loc}(L) \cap S$ defined by the orbits near $\Gamma \backslash U_0$.

In [3], Shilnikov used codings with infinitely many symbols, which he called “Birkhoff−Morse”, cf. [19]. Since the symbol 1 is followed by a string of zeros, any orbit in U, which is not asymptotic to L, may be coded by the infinite sequence

$$(\ldots, k_{-i}, \ldots, k_0, k_1, \ldots, k_i, \ldots),$$

where k_i is the number of zeros following the corresponding symbol 1. An orbit, which is asymptotic to L in the forward or backward time, is represented by a sequence that is infinite only in one direction:

$$[k_0,\, k_1, \ldots, k_i, \ldots) \;\text{ or }\; (\ldots, k_{-i}, \ldots,\, k_0]$$

respectively, whereas an orbit homoclinic to L is coded by the finite sequence

$$[k_0, k_1, \ldots, k_i],$$

where $i + 1$ equals the number of times the homoclinic orbit leaves U_0.

[4)] He kept a manuscript with the full proofs. We published it in the book of selected works of L. P. Shilnikov [27].

Remark 1. For the study of a neighborhood of a transverse homoclinic there is no essential difference between the $0-1$ code and this infinite-symbol code, but the latter has been found very useful in other problems, such as a homoclinic loop to a saddle-focus [46–48] and a homoclinic tangency [45, 49, 50].

Let $\{M_i\}$ be the sequence of intersections of an orbit Λ with the small neighborhood Π^+ of M^+. By construction,

$$M_{i+1} = T_1 T_0^{k_i - p+1} M_i, \tag{2.1}$$

where $[k_i]$ is the "Birkhoff – Morse" code of Λ (in the continuous-time case, take $p = 1$).

Thus, the proof of the Shilnikov Theorem reduces to showing that any code with sufficiently large k_i produces a unique sequence of points M_i satisfying (2.1). This requires analysis of the maps T_0^k, which is described in the next section.

2.2. The Shilnikov Boundary-Value Problem

The analysis of the long-time behavior near a saddle periodic orbit — the uniform estiamates on the maps T_0^k for arbitrarily large k — is easy in the case where the system near the periodic orbit is linear. However, the existence of a local smooth linearization is too strong a requirement.

Without the linearization, the problem is that small corrections to the solution due to nonlinear terms can lead to the exponential divergence with the solution of the linearized system. This causes the absence of uniform estimates for the solutions of the initial-value problem as the time the solution spends in a neighborhood of the saddle tends to infinity (i. e., the initial point gets closer to the stable manifold). To overcome this difficulty, Shilnikov introduced a special *boundary value problem* such that its solutions remain uniformly bounded along with the derivatives.

Consider a diffeomorphism T_0 near a saddle fixed point:

$$\bar{x} = Ax + f(x, y), \ \bar{y} = By + g(x, y). \tag{2.2}$$

Here, the eigenvalues of matrices A and B are assumed to be less and greater than 1 in the absolute value, respectively; the functions f and g are nonlinearities, i. e., they vanish at zero along with the first derivatives. The Shilnikov boundary-value problem is this:

given sufficiently small x_0, y_k, and any integer $k \geqslant 0$, to find x_k and y_0 such that $(x_k, y_k) = T_0^k(x_0, y_0)$.

As shown in [7, 45], see also [**?**], the unique solution exists and is uniformly bounded (along with the derivatives) for all $k \geqslant 0$. For a proof, one notices that $(x_j, y_j) = T_0^j(x_0, y_0)$, $j = 1, \ldots, k$, satisfy

$$\begin{aligned} x_j &= A^j x_0 + \sum_{i=0}^{j-1} A^{j-i-1} f(x_i, y_i), \\ y_j &= B^{j-k} y_k - \sum_{i=j}^{k-1} B^{j-i-1} g(x_i, y_i), \qquad j = 0, 1, \ldots, k. \end{aligned} \tag{2.3}$$

Next, one checks that the right-hand side acts as a contraction operator on the space of sequences $\{(y_0, x_1, y_1, \ldots, x_{k-1}, y_{k-1}, x_k\}$ for all small x_0, y_k.

With an appropriate choice of coordinates (x, y) (when the map T_0 is brought to the so-called main normal form [51]), system (2.3) for the solution of the boundary-value problem allows for very precise estimates for the trajectories near saddle periodic orbits [45, 50, 51]. Without normal form transformations, only assuming that the coordinates are chosen such that the local stable and unstable manifolds are straightened, $W^s_{loc} : y = 0$ and $W^u_{loc} : x = 0$, one can show that

$$\|(x_k, y_0)\|_{C^1} = o(1)_{k\to\infty}, \tag{2.4}$$

(see e. g. [7]).

Originally, in [3], Shilnikov considered the continuous-time case. He writes the system near a saddle periodic orbit in the form

$$\dot{x} = Ax + f(x, y, t), \quad \dot{y} = By + g(x, y, t), \tag{2.5}$$

where the matrices A and B are such that their eigenvalues have negative and positive real parts, respectively. The functions f and g are periodic in t and vanish along with their derivatives at $(x = 0, y = 0)$ (thus, the origin $(x = 0, y = 0)$ corresponds to the saddle periodic orbit). The solution of the boundary-value problem $(x(0) = x_0, y(\tau) = y_1)$ satisfies the system of integral equations

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)} f(x(\tau), y(\tau), \tau)\, d\tau, \\ y(t) &= e^{B(t-t_0)}y_1 + \int_t^{t_0} e^{B(t-\tau)} g(x(\tau), y(\tau), \tau)\, d\tau. \end{aligned} \tag{2.6}$$

Shilnikov proved that this system has a unique solution, uniformly bounded for all $\tau \geqslant 0$. Beyond the Poincaré – Birkhoff problem, these integral equations have found numerous applications in the study of bifurcations of homoclinic loops of saddle equilibrium states (in this case f and g in (2.6) are independent of t), see [25, 46**?**].

2.3. Method of Cross Maps: Saddle Fixed Points of Operator Sequences

Another key ingredient in the Poincaré – Birkhoff problem paper [3] were *theorems on the existence and uniqueness of fixed points for sequences of operators* acting on countable products of metric spaces, see Theorems A and B below.

Consider countably many complete metric spaces Z_i, $i = 0, \pm 1, \ldots$, with metrics $\rho_i(z_i', z_i'')$. Let operators C_i be such that

$$C_i Z_i \subset Z_{i+1}, \quad \rho_{i+1}(C_i z_i', C_i z_i'') < q\rho_i(z_i', z_i''),$$

where the constant q is independent of i. We assume also that the diameter of the spaced Z_i is uniformly bounded:

$$\sup_{z_i', z_i'' \in Z_i} \rho_i(z_i', z_i'') < d < \infty.$$

Consider the direct product $Z = \prod_{i=-\infty}^{+\infty} Z_i$, the space of sequences $z = (\ldots, z_-, \ldots, z_{-1}, z_0, z_1, \ldots, z_i, \ldots)$. The distance in Z is defined as

$$\rho(z', z'') = \sup_{-\infty < i < +\infty} \rho_i(z_i', z_i'').$$

Consider the operator C: $z \xrightarrow{A} \bar{z}$, where $\bar{z}_{i+1} = C_i z_i$ for all i.

Theorem A. *If $q < 1$, then C is contracting, so there exists a unique sequence $z^* \in Z$ such that $z^*_{i+1} = C_i z_i*$ for all i.*

Obviously, for the existence of the unique sequence z^* it is sufficient to require the contraction of the operators A_i only for all $i < 0$. Namely, Theorem A can be formulated for one-sided sequences as follows:

If the operators C_i are uniformly contracting for all $i < 0$, then there exists a unique sequence $z^* = (\ldots, z^*i, \ldots, z^*_0)$ such that $z^*_i = C_{i-1} z_{i-1}*$ for all $i \leqslant 0$.

Let us now define the notion of a saddle operator. Consider a pair of metric spaces (X, Y) and a pair of metric spaces $(\bar{X}, \bar{Y})$. We say that an operator T is represented in the *cross form* if there exists the cross map $T^\times : (X, \bar{Y}) \to (\bar{X}, Y)$ such that $(\bar{x}, \bar{y}) = T(x, y)$ if and only if $(\bar{x}, y) = T^\times(x, \bar{y})$. The operator T is called *saddle* if its cross map $T^\times$ is contracting.

Consider now two sequences of complete metric spaces X_i and Y_i $(i = 0, \pm 1, \dots)$ such that

$$\sup_{x_i', x_i'' \in X_i} \rho_{X_i}(x_i', x_i'') < d, \qquad \sup_{y_i', y_i'' \in Y_i} \rho_{Y_i}(y_i', y_i'') < d,$$

where the constant d is independent of i. Let operators A_i, B_i be such that

$$A_i(X_i \times Y_{i+1}) \subset X_{i+1}, \ \ B_i(X_i \times Y_{i+1}) \subset Y_i.$$

Suppose that for some constant q, independent of i, we have

$$\begin{aligned} \rho_{X_{i+1}}(\bar{x}_{i+1}', \bar{x}_{i+1}'') &< \frac{q}{2}\left(\rho_{X_i}(x_i', x_i'') + \rho_{Y_{i+1}}(y_{i+1}', y_{i+1}'')\right), \\ \rho_{Y_i}(\bar{y}_i', \bar{y}_i'') &< \frac{q}{2}\left(\rho_{X_i}(x_i', x_i'') + \rho_{Y_{i+1}}(y_{i+1}', y_{i+1}'')\right), \end{aligned} \tag{2.7}$$

where $\bar{x}_{i+1} = A_i(x_i, y_{i+1}),\ \bar{y}_i = B_i(x_i, y_{i+1})$.

Applying Theorem A to the sequence of spaces $Z_i = (X_i \times Y_i)$ and operators $C_i = (A_i, B_i)$, we obtain

Theorem B. *Let $q < 1$. Then, there exists a unique sequence*

$$(\dots, (x_{-1}^*, y_{-1}^*), (x_0^*, y_0^*), (x_1^*, y_1^*) \dots),$$

which, for all i, satisfies the conditions

$$x_{i+1}^* = A_i(x_i^*, y_{i+1}^*), \ \ y_i^* = B_i.(x_i^*, y_{i+1}^*).$$

The condition $q < 1$ means that the maps T_i whose cross form is given by $T_i^\times = (A_i, B_i)$ are saddle operators. Thus, Theorem B gives us the existence of a unique bounded orbit of a sequence of saddle operators.

Since return maps near a transverse homoclinic orbit are saddle (see the next Section), Theorem B provides a tool for the proof of the existence and uniqueness of an orbit in the neighborhood U any given natural coding, the main claim of the Shilnikov Theorem.

One can use Theorem A in order to prove, in the spirit of J. Hadamard ideas [52], the existence of stable and unstable manifolds for the trajectory of the sequence of saddle operators for that is given by Theorem B. For that, define Z_i as the space, with the C^0-metrics, of Lipshitz functions $x_i = p_i(y_i)$ for an appropriate Lipshitz constant. As C_i take the corresponding graph transform operators: one can show that these operators are uniformly contracting, so Theorem A gives the existence of a Lipshitz unstable manifold. The stable manifold follows from the symmetry of the problem. This technique was used e. g. in the study of the Lorenz attractor in [64].

2.4. Completion of the Proof: Reduction to Theorem B

Now we can complete the proof of the Shilnikov Theorem. The original proof in [3] was done for the continuous-time case. We choose to present the proof in the case of discrete-time. There is no principal difference: in both cases the problem is reduced to the proof that the return maps to a small neighborhood of the homoclinic point M^+, see Section 2.1, are uniformly saddle (i. e., the corresponding constant q in (2.7) is the same for all maps). For that, Shilnikov studied in [3] the solution of the boundary-value problem (2.6). We, here, deal with the boundary-value problem (2.3), following [7, 45].

We consider a diffeomorphism T with a saddle periodic point O and a transverse homoclinic orbit Γ (Fig. 2a). Let us choose coordinates (x, y) in a small neighborhood U_0 of O such that the local stable and unstable invariant manifolds W^s_{loc} and W^u_{loc} of O take the form $y = 0$ and $x = 0$, respectively, i. e., they are straightened. This means that the local map T_0 takes the form

$$\bar{x} = Ax + f(x, y),\ \bar{y} = By + g(x, y),$$

where $\|A\| < 1, \|B^{-1}\| < 1$, and

$$f(0,y) \equiv 0, \quad g(x,0) \equiv 0.$$

Take a pair of homoclinic points $M^+(x^+,0) \in W^s_{loc}, M^-(0,y^-) \in W^u_{loc}$ and choose sufficiently small neighborhoods Π^+ and Π^- of M^+ and M^- such that $T_0(\Pi^+) \cap \Pi^+ = \varnothing, T_0^{-1}(\Pi^-) \cap \Pi^- = \varnothing$. Then the set of points from Π^+ that hit Π^- at iterations of the local map T_0 consists of countably many disjoint strips $\sigma_k^+ \subset \Pi^+,\ j = \bar{n}, \bar{n}+1, \ldots$. In the same way, a countable set of strips $\sigma_k^- = T_0^j \sigma_k^+ \subset \Pi^-$ is defined in Π^-, see Fig. 5a. The strips σ_k^+ and σ_k^- accumulate to W^s_{loc} and W^u_{loc} as $k \to \infty$.

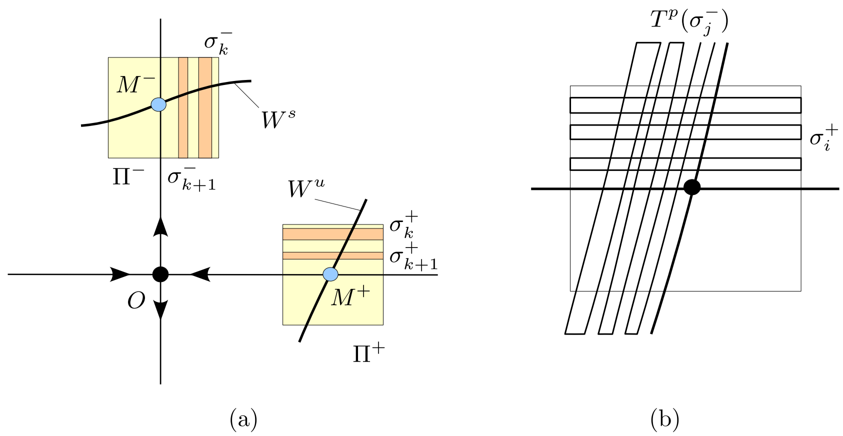


Fig. 5. (a) strips σ_k^- and σ_k^+ in U_0; b) because of the transversality of $T_1(W^u_{loc})$ with $W^s w_{loc}$ the image of each vertical strip $(T_1(\sigma_j^-))$ intersects every horizontal strip (σ_i^+).

The map $T_0^k : \sigma_k^+ \to \sigma_k^-$ can be written in the cross form

$$x_k = \xi_k(x_0, y_k), y_0 = \eta_k(x_0, y_k), \tag{2.8}$$

where $(x_0, y_0) \in \sigma_k^+, (x_k, y_k) \in \sigma_k^-$. This is given by the solution of the system (2.3). Since the local manifolds are straightened, we have estimate (2.4) which can be written as

$$\|\xi_k\|_{C^1} + \|\eta_k\|_{C^1} = o(1)_{k\to\infty}. \tag{2.9}$$

Let us write the global map $T_1 : \Pi^- \to U_0$ in the form

$$\bar{x} - x^+ = F(x, y - y^-), \quad \bar{y} = G(x, y - y^-), \tag{2.10}$$

where $F(0,0) = 0, G(0,0) = 0$ since $M^+ = T_1 M^-$. The transversality of $T_1(W^u_{loc})$ to W^s_{loc} at the point M^+ means that $\frac{\partial G}{\partial y}|_{M^-}$ is invertible. Therefore,

$$\left\| \frac{\partial G}{\partial y}^{-1} \right\| < C < \infty. \tag{2.11}$$

Let now Λ be an orbit of diffeomorphism T lying entirely in U. Suppose Λ is not asymptotic to O, then it visits Π^+ infinitely many times and its intersection points with Π^+ must belong to the strips σ_k^+. Let $M_i(x_0^i, y_0^i) \in \sigma_{k_i}^+,\ i = 0, \pm 1, \ldots,$ be the consecutive intersection points. Then $M_i^1(x_1^i, y_1^i) = T_0^{k_i} M_i \in \sigma_{k_i}^-$ are the successive intersection points of Λ with Π^-. It is evident that $M_{i+1} = T_1(M_i^1)$.

From (2.8) and (2.10) we derive the following relations

$$x_0^{i+1} - x^+ = F(\xi_{k_i}(x_0^i, y_1^i), y_1^i - y^-),$$
$$\eta_{k_{i+1}}(x_0^{i+1}, y_1^{i+1}) = G(\xi_{k_i}(x_0^i, y_1^i), y_1^i - y^-),$$
$$i = 0, \pm, \dots .$$

By the transversality condition (2.11) and estimate (2.9), this system can be rewritten in the cross form

$$\begin{aligned} x_0^{i+1} - x^+ &= \tilde{F}_i(x_0^i, y_1^{i+1}), \\ y_1^i - y^- &= \tilde{G}_{i+1}(x_0^i, y_1^{i+1}), \\ i &= 0, \pm, \dots . \end{aligned} \tag{2.12}$$

Moreover, by (2.9),

$$\|(\tilde{F}_i, \tilde{G}_i\|_{C^1} = o(1)_{k_i, k_{i+1} \to \infty}.$$

Now, for $\bar{n}$ large enough, the existence of a unique solution $\{(x_0^i, y_1^i)\}, i = 0, \pm 1 \dots$, of system (2.12) is given by Theorem B of Section 2.3, where one chooses the spaces and operators as follows: $X_i = \{\|x_0^i - x^+\| < \varepsilon\}, Y_i = \{\|y_1^i - y^-\| < \varepsilon\}$ and $A_i = \tilde{F}_i, B_i = \tilde{G}_i$.

With obvious modifications, the proof of the existence of asymptotic to O orbits is done in the same way.

Note that the case of a transverse heteroclinic cycle was also considered in [3]. A heteroclinic cycle is an invariant set consisting of several saddle periodic orbits and orbits of transverse intersection of their stable and unstable manifolds. Shilnikov gave a complete description of the set of all orbits from a small neighborhood of such cycle. The result was later generalized in papers with V. S. Afraimovich [7, 8].

3. HOMOCLINIC TORUS

Soon after the paper on the Poincaré – Birkhoff problem, Shilnikov published a short paper about homoclinics to invariant tori [25]. The complete proof of the results was published in [26]. The theory of normal hyperbolicity [40, 41] did not exist at the time, so the original formulation in [25] included assumptions on the map near a saddle invariant torus that simplified the presentation of the results (the function h in (3.1) below was assumed zero). The manuscript [26] with the proofs did not make this assumption, nor relied on it. This made the formulation of the conditions of the theorems (as well as proofs) quite heavy. We use below an equivalent, but more geometric description.

Let T be an $(l + m + n)$-dimensional diffeomorphism that has an l-dimensional invariant torus τ. Let U_0 be a sufficiently small neighborhood of τ. Assume that the map $T|_{U_0}$ can be written in the following form:

$$\begin{aligned} &\bar{x} = A(\theta)x + f(x, y, \theta), \qquad \bar{y} = B(\theta)y + g(x, y, \theta), \\ &\bar{\theta} = \Psi(\theta) + h(x, y, \theta), \end{aligned} \tag{3.1}$$

where $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_n)$, $\theta = (\theta_1, \dots, \theta_l)$, and the smooth functions A, B, f, g, h and Ψ are periodic in θ_s with period 1, for each $s = 1, \dots, l$.

Assume that the functions f, g, h, along with their first derivatives, vanish at $x = 0$, $y = 0$. This means that the invariant torus τ is given by the equation $(x = 0, y = 0)$. Let

$$\sup_\theta \|A(\theta)\| < 1, \quad \sup_\theta \|B^{-1}(\theta)\| < 1, \tag{3.2}$$

so τ is a saddle torus. Assume that

$$\sup_\theta \|A(\theta)\| \cdot \sup_\theta \left\| \frac{\partial \Psi}{\partial \theta} \right\| \cdot \sup_\theta \left\| \left(\frac{\partial \Psi}{\partial \theta} \right)^{-1} \right\| < 1, \quad \sup_\theta \|B^{-1}(\theta)\| \cdot \sup_\theta \left\| \frac{\partial \Psi}{\partial \theta} \right\| \cdot \sup_\theta \left\| \left(\frac{\partial \Psi}{\partial \theta} \right)^{-1} \right\| < 1, \tag{3.3}$$

which implies that the torus τ is *normally hyperbolic* and, thus, is preserved for smooth small perturbations [40].

Note that since $T|_\tau : \theta \mapsto \Psi(\theta)$ is a diffeomorphism, it follows that $\sup_\theta \|\frac{\partial\Psi}{\partial_\theta}\| > 1$ and $\sup_\theta \|(\frac{\partial\Psi}{\partial_\theta})^{-1}\| > 1$, so conditions (3.3) imply (3.2).

Suppose that the intersection of the stable and unstable manifolds $W^s(\tau)$ and $W^u(\tau)$ contains an l-dimensional torus γ. We assume that $W^s(\tau)$ and $W^u(\tau)$ are transverse at each point $M \in \gamma$, i. e., the intersection of the tangent spaces of $W^s(\tau)$ and $W^u(\tau)$ at each point $M \in \gamma$ is R^l.

The next assumption is that

the homoclinic torus γ is transverse to the strong-stable invariant foliation in $W^s(\tau)$ and the strong-unstable invariant foliation in $W^u(\tau)$.

Since γ is a homoclinic torus, it means that its orbit — the sequence of tori $\gamma_i = T^i\gamma$, $i = 0, \pm1, \pm2, \ldots$ — tends to τ both at forward and backward iterations of T. Following Shilnikov, we call the sequence $\Gamma = (\ldots, \gamma_{-1}, \gamma_0 = \gamma, \gamma_1, \ldots, \gamma_i, \ldots)$ a *homoclinic tube* of the torus τ, see Fig. 6 for an illustration.

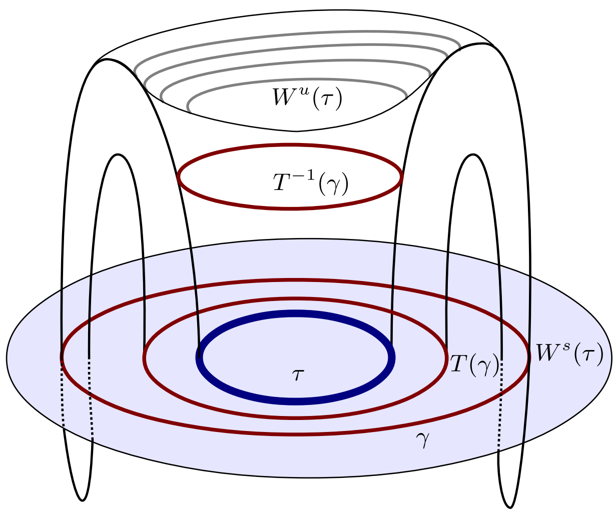


Fig. 6. A part of a homoclinic tube Γ of the torus τ is shown for the case where T is a map of R^3.

Let $\gamma^- \in W^u_{loc}(\tau)$ and $\gamma^+(\tau) \in W^s_{loc}(\tau)$ be a pair of homoclinic tori from the homoclinic tube Γ. By assumption, γ^- and γ^+ are transverse to the strong-unstable and strong-stable foliations $\mathcal{F}^{uu}_{loc}$ and $\mathcal{F}^{ss}_{loc}$ in $W^u_{loc}(\tau)$ and $W^s_{loc}(\tau)$, respectively. Assume that

the homoclinic torus γ^- intersects each leaf of $\mathcal{F}^{uu}_{loc}$ exactly once, and the homoclinic torus γ^+ intersects each leaf of $\mathcal{F}^{ss}_{loc}$ exactly once.

Along with the transversality, this condition means that the projection map of γ^- to τ by the leaves of $\mathcal{F}^{uu}_{loc}$ and the projection map of γ^+ to τ by the leaves of $\mathcal{F}^{ss}_{loc}$ are diffeomorphisms. One can always introduce coordinates in the neighborhood of τ such that the local stable and unstable manifold, along with their strong-stable and strong-unstable foliations are straightened, i. e.,

$$f(0, y, \theta) \equiv 0, \qquad g(x, 0, \theta) \equiv 0, \qquad h(0, y, \theta) \equiv 0, \qquad h(x, 0, \theta) \equiv 0$$

in (3.1). With this choice of coordinates, the above conditions can be formulated as follows:

the homoclinic tori γ^+ and γ^- are graphs of smooth functions of θ:

$$\gamma^+ : \{x = x^+(\theta),\ y = 0\}, \qquad \gamma^- : \{y = y^-(\theta),\ x = 0\}.$$

Let U be a sufficiently small neighborhood of the homoclinic tube $\Gamma \cap \tau$. It consists of a solid torus U_0 around τ and a finite number of solid tori around the homoclinic tori of $\Gamma \setminus U_0$. The sequence $\{\tilde{\gamma}_i\}$, $i = 0, \pm 1, \pm 2$, comprised of l-dimensional tori $\tilde{\gamma}_i \subset U$ such that $\tilde{\gamma}_{i+1} = T(\tilde{\gamma}_i)$, is called an *invariant tube*. Similarly to the natural coding in a neighborhood of a homoclinic to a periodic orbit (see Section 2.1), the invariant tubes that lie in U can be symbolically represented using sequences of 0's and 1's:

$$[\ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots],$$

where 1's correspond to those tori in the tube, which lie in $U \setminus U_0$ but their image lies in U_0. Let $\Omega_{\bar{n}}$ be the set of all sequences of 0's and 1's such that each 1 is followed by a sequence of 0's of the length at least $\bar{n}$.

Theorem 2 ([25]). *The neighborhood U can be chosen such that the set N of all points whose orbits lie entirely in U is a union of normally-hyperbolic (saddle) invariant tubes. For a sufficiently large integer $\bar{n}$, for any sequence $\omega = \Omega_{\bar{n}}$ there exists exactly one invariant tube in N with the code ω. Periodic codes correspond to periodic tori, homoclinic codes (with infinite tails of 0's on both sides) correspond to homoclinic tubes of τ.*

Importantly, it was noted in [25] that the theorem remains valid for an arbitrary smooth invariant manifold τ (with obvious modifications). For example τ can be R^1, $R^1 \times S^1$, etc. Indeed, the assumption that τ is a torus was not used in the proof in [26].

4. NONAUTONOMOUS HOMOCLINIC STRUCTURES

The study of homoclinic dynamics in general non-autonomous systems was started by Shilnikov almost immediately after his works [3, 25]. He posed (in a hand-written note) the corresponding problem to L. M. Lerman in 1970 as one of the main topics of his PhD thesis [42], see [43].

As a motivation, one can consider a small time-dependent perturbation of an autonomous system in a plane:

$$\dot{x} = F_0(x) + \varepsilon F_1(x, t),$$

where the unperturbed system $\dot{x} = F_0(x)$ has a hyperbolic saddle equilibrium $x = 0$ with a homoclinic loop. In the extended phase space (x, t), the line $L : x(t) = 0$ is a hyperbolic integral curve at $\varepsilon = 0$, and its stable and unstable integral manifolds coincide (entirely — in the case of a homoclinic figure-eight, or half — in the case of a single loop); so, all integral curves in them are homoclinic to the saddle line L. At $\varepsilon \neq 0$, the saddle integral curve L persists. The integral manifolds $W^s(L)$ and $W^u(L)$ also persist, but they can split. For a general perturbation, they intersect only over a discrete set of transverse homoclinic curves. When this set is non-empty and infinite, we obtain a nonautonomous analogue of the Poincaré–Birkhoff problem.

To explain the statement of the problem, let us first revisit the time-periodic case. Consider a smooth nonautonomous system

$$\dot{x} = F(x, t),$$

where $x \in R^n$ and $F(x, t)$ is a T-periodic function of t, i. e., $F(x, t + T) \equiv F(x, t)$. Let the system have a T-periodic solution $x = p(t)$. Let us assume that it is hyperbolic, i. e., it has n multipliers, k of which lie inside and $n - k$ outside the unit circle. Then, in the extended phase space $R^n \times R$, a pair of smooth invariant submanifolds W^s and W^u, of dimensions $k + 1$ and $n - k + 1$, will intersect transversely along the integral curve $L : (p(t), t)$. These manifolds are periodic sets in $R^n \times R$, i. e., they are invariant under the shifts $(x, t) \to (x, t + iT), i \in Z$. Assume that W^s and W^u also intersect each other along integral curves (other than L) that are homoclinic, i. e., asymptotic to L as $t \to \pm\infty$.

Due to the periodicity of the system, if one such homoclinic solution Γ_0 exists, then $W^s \cap W^u$ contains infinitely many such solutions Γ_n; they are obtained from Γ_0 by the iT-shifts along the t-axis in $R^n \times R$. If the intersection of W^s and W^u along Γ_0 is transverse, the same holds for all Γ_i.

In the periodic case, the problem of the description of the dynamics in a neighborhood of the set of homoclinic solutions is reduced to the study of the autonomous system $\bar{x} = F(x, \theta), \dot{\theta} = 1$, where $\theta \in S^1$ is a cyclic coordinate. In this case, the homoclinic curves Γ_i are glued together into one transverse homoclinic orbit, as in Fig. 1. One can also consider the Poincaré period map $\{t = 0\} \to \{t = T\}$, see Fig. 2. In any case, the Shilnikov Theorem is automatically applied.

None of this works when the general non-autonomous systems are considered. However, one can identify two basic properties that allow one to describe the homoclinic chaos in non-periodic systems. These properties are:

1) the existence of an infinite set $\mathcal{G} = \{\Gamma_i\}_{i\in Z}$ of integral homoclinic curves double-asymptotic to an integral curve L, see Fig. 7 and

2) the existence of a uniform exponential dichotomy as $t \to \pm\infty$ for L and for all $\Gamma_i \in \mathcal{G}$.

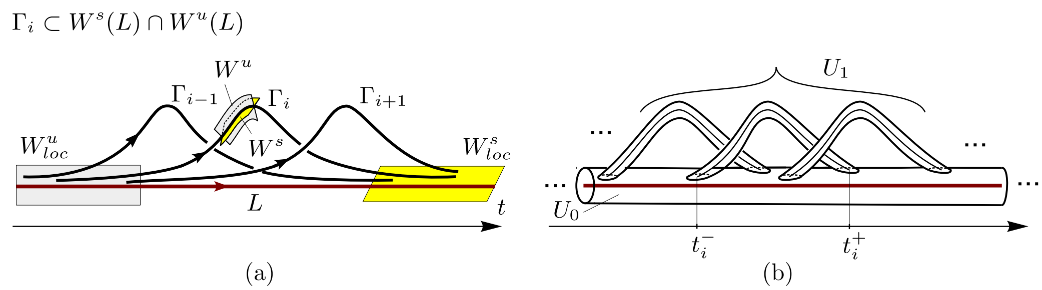


Fig. 7.

Let us consider a general nonautonomous system

$$\dot{x} = F(x, t)$$

in the extended phase space $R^n \times R^1$. Assume that the derivative $\partial_x F$ is uniformly bounded and uniformly continuous.

Definition 1. An integral curve $L : x = p(t)$ is called hyperbolic if it satisfies a uniform exponential dichotomy. Namely, the linearized system $\dot{x} = \partial_x F(p(t), t) \cdot x$ has two invariant and uniformly continuous families of subspaces $E^s(t)$ and $E^u(t)$ such that $E^s(t) \oplus E^u(t) = R^n$ for each $t \in R^1$ and the angle between $E^s(t)$ and $E^u(t)$ is bounded away from zero. Moreover, there are positive constants C and λ such that

$$\begin{aligned} &||\mathcal{T}(t,\tau)\ \xi|| \leqslant C \exp[-\lambda(t-\tau)]\ ||\xi|| \ \text{ for all } \ t \geqslant \tau \ \text{ and } \ \xi \in E^s(\tau), \\ &||\mathcal{T}(t,\tau)\ \eta|| \leqslant C \exp[-\lambda(\tau - t)]\ ||\eta|| \ \text{ for all } \ t \leqslant \tau \ \text{ and } \ \eta \in E^u(\tau), \end{aligned} \tag{4.1}$$

where $\mathcal{T}(t, \tau)$ is the fundamental matrix of the linearized system from time τ to time t.

Definition 2. A set of integral curves is uniformly hyperbolic if every curve of the set is hyperbolic, the subspaces E^s and E^u uniformly continuously depend on the points in the set, and the constants C and λ in (4.1) can be chosen to be the same for all curves of the set.

By the Hadamard – Perron theorem, a hyperbolic integral curve L has, in its uniform cylindrical neighborhood, smooth local stable and unstable manifolds $W^s_{loc}(L)$ and $W^u_{loc}(L)$ which are tangent to E^s and E^u, respectively. These local invariant manifolds are extended by the nonautonomous flow to form the global stable $W^s(L)$ and unstable $W^u(L)$ manifolds. Suppose they intersect transversely over a countable set $\mathcal{G} = \{\Gamma_i\}_{i\in Z}$ of integral curves. The transversality, along with the exponential dichotomy for L, implies that the angle between $E^s(t)$ and $E^u(t)$ is bounded away from zero for all t. Denote the minimal angle as α_i and assume that

(*i*) *the transversality is uniform*:

$$\inf_{\Gamma_i \in \mathcal{G}} \alpha_i > 0.$$

Let us choose a sufficiently thin, uniform cylindrical neighborhood $U_0(L)$. Let the boundary of U_0 contain cross-sections S^+ to $W^s_{loc}(L)$ and S^- to $W^u_{loc}(L)$. If they are sufficiently close to L, then for each homoclinic curve $\Gamma_i \in \mathcal{G}$ there exist two moments of time t_i^+ and t_i^- corresponding to intersections with S^+ and S^-, respectively, such that Γ_i stays outside U_0 only for $t \in [t_i^+, t_i^-]$. Thus, t_i^- is the exit time from U_0 and t_i^+ is the time of re-entry, Fig 7b. Assume that

(*ii*) *the transition times are uniformly bounded*:

$$t_i^+ - t_i^- \leqslant D,$$

where the constant D is the same for all Γ_i.

Note that this condition holds true if we take a thinner neighborhood U_0, just the constant D will increase. One can show that the conditions (*i*), (*ii*) imply that the set $L \cup \mathcal{G}$ is uniformly hyperbolic.

Under the above conditions, the main result of [43] is the following theorem. Let $U = U_0 \cup U_1$, where U_1 is a small neighborhood of $\mathcal{G} \backslash U_0$ (the set U_1 is a union of a countable set of thin handles attached to U_0, Fig 7b).

Theorem 3. *For any sufficiently large $T > 0$, the neighborhood U of $L \cup \mathcal{G}$ can be chosen such that the set of all integral curves lying entirely in U is in one-to-one correspondence with set of all, finite or infinite, sequences of integers i_k satifying the following condition:*

$$t^-_{i_{k+1}} - t^-_{i_k} \geqslant T$$

for all k.

In this theorem, the coding sequence $[i_k]$ corresponds to an integral curve that for $t \in (t^-_{i_k}, t^+_{i_k})$ stays close to Γ_{i_k}, and outside these time intervals it stays close to L. If the coding sequence is finite, it corresponds to an integral curve homoclinic to L. Sequences finite from the left correspond to integral curves asymptotic to L as $t \to -\infty$ (i. e., they belong to $W^u(L)$). If the coding sequence is finite from the right, then the corresponding integral curve is asymptotic to L as $t \to +\infty$ (i. e., it belongs to $W^s(L)$). The empty sequence corresponds to L, by definition.

Partial cases of this theorem (for almost periodic systems) were considered in [53–55]. For applications of this theorem in physics see [56, 57].

5. INFINITE-DIMENSIONAL HOMOCLINICS

In the 70s, after publication of papers by Lorenz [58], Ruelle and Takens [59], and many other physicists, the hope rose for the explanation of hydrodynamical turbulence based on the new-born chaos theory. This caused a surge of interest to the complicated dynamics of infinite-dimensional systems, mainly those which are generated by partial differential equations. For Shilnikov, the problem of turbulence was among motivations behind his work on the Lorenz attractor [63, 64], spiral chaos [48, 60], and the torus-chaos transition [61, 62].

At the same time, Shilnikov set up the general problem of extension of what was known about homoclinic structures onto infinite-dimensional systems. Along with Lerman, they decided to study smooth maps of a Banach space which have a periodic point with a transverse homoclinic orbit. The results — the infinite-dimensional analogue of the Shilnikov theorem — were published in [**?**].

The main difficulty of the infinite-dimensional case is that the map can be non-invertible, as it is typical, for example, for the time-1 maps generated by parabolic PDEs. In such case, the hyperbolicity and transversality are not enough. Therefore, in [**?**] additional conditions are imposed.

Let O be a fixed point of a smooth map f of a Banach manifold B. Denote $L = Df(O) : \mathcal{T}_O B \to \mathcal{T}_O B$, i. e., L is the Frechet derivative of f at O (here $\mathcal{T}_P B$ denotes the tangent space to B at a point P). Suppose

A) *O is a hyperbolic saddle,*

i. e., the spectrum $Spec(L)$ of the bounded linear operator L does not intersect the unit circle and breaks into two closed nonempty spectral sets σ_s and σ_u that lie, respectively, inside and outside the unit circle. The hyperbolicity implies that $\mathcal{T}_O B$ decomposes into direct sum of two invariant subspaces: $\mathcal{T}_O B = E_s \oplus E_u$, where $Spec(L|_{E_s}) = \sigma_s, Spec(L|_{E_u}) = \sigma_u$.

Thus, in a neighborhood U_0 of the fixed point O, one can use coordinates $(x, y) \in E_s \oplus E_u$ such that the local map $T_0 = f|_{U_0}$ can be written in the form

$$(x, y) \to \big(L_s x + p(x, y), L_u y + q(x, y)\big), \tag{5.1}$$

where $L_s = L|_{E_s}, L_u = L|_{E_u}$, and p and q are smooth functions vanishing at zero along with the first derivatives. Without loss of generality, we can assume $||L_s|| < 1, ||L_u^{-1}|| < 1$.

Since $Spec(L_u)$ is bounded away from zero, it follows that L_u is an invertible operator. However, the operator L_s can be non-invertible. Nevertheless, one can show that the map T_0 given by (5.1) is a saddle map, as described in Section 2.3. Thus, the Hadamard – Perron theorem is applicable. It implies that the local coordinates can be chosen such that $x = 0$ is the local unstable manifold W^u_{loc} and $y = 0$ is the local stable manifold W^s_{loc}. The manifold W^s_{loc} is forward invariant and consists of all points of U_0 whose forward orbits by T_0 tend to O. The restriction $T_0|_{W^u_{loc}}$ is invertible; the manifold W^s_{loc} is backward invariant and consists of all points of U_0 whose backward orbits by T_0 tend to O.

Iterations of f extend W^u_{loc} to the global unstable manifold W^u. Assume that W^u intersects W^s_{loc} at a homoclinic point M^+. So, there exists a homoclinic orbit Γ, bi-asymptotic to O. Due to the possible non-invertibility of f, the point M^+ can have several backward orbits. We choose just one homoclinic orbit through M^+ and study the orbits in its small neighborhood U.

A subtlety in the infinite-dimensional case is the notion of transversality of submanifolds in a Banach space. Recall that, in a finite-dimensional manifold, if two smooth submanifolds of the complementary dimensions intersect transversely at some point, then the intersection point is isolated. In the infinite-dimensional Banach manifold this is no longer true: the dimension of intersection of transverse submanifolds is indetermined (examples of intersections over a manifold of any finite or infinite dimension were presented in [**?**]).

Therefore, the transversality condition was replaced in [**?**] by the requirement that

B) *the tangent spaces to W^u and W^s_{loc} at the homoclinic point M^+ are complementary:*

$$\mathcal{T}_{M^+} W^u \oplus \mathcal{T}_{M^+} W^u = \mathcal{T}_{M^+} B.$$

One can show that if $\dim W^u < \infty$ (like in the case of parabolic PDEs or delay equations), this condition is equivalent to the transversality of W^u and W^s_{loc} [**?**].

Remind that the map f can be non-invertible. Thus, not every point can have a pre-image, while some points can have several preimages. The following condition excludes the last possibility:

C) *for any $x \in B, y \in B$, if $x \neq y$, then $f(x) \neq f(y)$.*

Note that this condition is fulfilled for parabolic PDEs and delay equations, so it is not very restrictive. Condition C does not prohibit the non-invertibility, so there still can be points which have no backward orbits, but if a backward orbit exists, it is unique.

We call an infinite sequence of points $\{M_i\}_{i=-\infty}^{+\infty}$ such that $M_{i+1} = f(M_i)$ for all i a *full orbit of the point M_0*.

Theorem 4. *Under conditions A, B, and C, the set N of all points, for which a full orbit exists and lies entirely in a small neighborhood U of the closure of the homoclinic orbit Γ, is in one-to-one correspondence with a set $\Omega_{\bar{n}}$ of all infinite sequences of 0's and 1's such that each 1 is followed by at least $\bar{n}$ of 0's, where $\bar{n}$ is a sufficiently large integer. The restriction of f to the set N is topologically conjugate to the shift on $\Omega_{\bar{n}}$.*

As one can see from the proof in [**?**], without condition C there still is a one-to-one correspondence (homeomorphism) between $\Omega_{\bar{n}}$ and the set $\hat{N}$ of all full orbits (with marked initial points) that lie entirely in U, see also [65, 66]. However, a homeomorphism between N and $\hat{N} \cong \Omega_{\bar{n}}$ may not exist without condition C, because of the possibility of orbits gluing together.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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