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Example of a Blue Sky Catastrophe

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To the memory of Professor E. A. Leontovich-Andronova

Abstract. We present a low-order system of ODEs exhibiting the blue sky catastrophe—a new codimension one bifurcation of periodic orbits.

1. Introduction

The known boundaries of stability (or existence) regions of periodic orbits in systems of differential equations can be classified by means of the following formal criterion: whether a bifurcating periodic orbit on the boundary exists or not. The codimension-one bifurcations listed below comprise the first group:

- A saddle-node (fold) bifurcation: two periodic orbits, one stable and one unstable, coalesce on the stability boundary and annihilate beyond it;
- Period-doubling (flip) bifurcation: one multiplier of the bifurcating periodic orbit equals -1;
- A bifurcation from a periodic orbit to a two-dimensional invariant torus; by rephrasing A. Andronov: "a cycle loses its skin".

All three cases above reduce to the analysis of stability of the corresponding fixed point of the Poincaré map on a cross-section transverse to the periodic orbit.

The codimension-one bifurcations of the second group have much in common with the situations in which a periodic orbit merges with an equilibrium state. The enumeration begins with the Andronov-Hopf bifurcation in which a periodic orbit shrinks into an equilibrium state with a pair of characteristic exponents $\pm i\omega$, $\omega > 0$. It should be noted that the period T of such bifurcating orbit can be estimated as $T \sim 2\pi/\omega$. In the other two cases the periodic orbit adheres to a homoclinic loop to an unstable equilibrium state which can be either a saddle with characteristic exponents in both open left and right half-planes, or a simple saddle-node with one zero real exponent. Since the vector field vanishes at the equilibrium state, the period of the bifurcating orbit tends to infinity as it approaches the homoclinic loop, while its perimeter remains of finite length.

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One may ask if the list of bifurcations and the associated stability boundaries is complete or not. This appeared to be the case until Shilnikov and Turaev [**1**] suggested a new scenario for the saddle-node bifurcation of periodic orbits, which may result in the appearance of a stable periodic orbit of an infinitely long period and infinitely large perimeter. Due to this feature, a virtual bifurcation of this kind was called, following R. Abraham, "a blue sky catastrophe". Moreover, this boundary may, under certain conditions, separate Morse–Smale systems from systems with hyperbolic attractors in the parameter space [**2**].

Here we present an example of a three-dimensional system in which the blue sky catastrophe develops in accordance with the phenomenological mechanism devised by Shilnikov and Turaev. Let us first discuss its geometrically comprehensive construction. The existence of the paper by Shilnikov and Turaev in this volume, to which the reader is referred to for a deeper insight, allows us to give only a brief explanation of the bifurcation setup.

It is assumed that there exists a saddle-node periodic orbit L^* whose unstable manifold $W^u_{L^*}$ returns to L^* as $t \to +\infty$, as shown in Figure 1; hence, the closure of the unstable manifold is not a Hausdorff manifold.

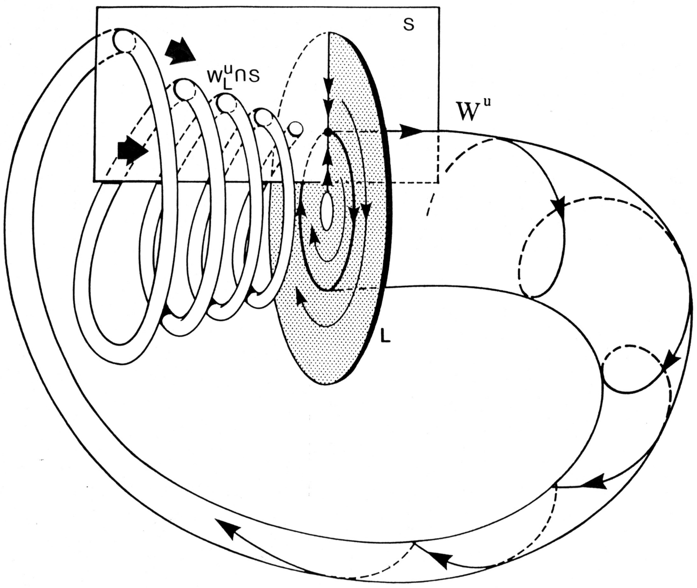

FIGURE 1. Geometrical interpretation of a blue sky bifurcation

The second assumption concerns the global property of the vector field; namely, the contraction of the phase volume in the direction transverse to the global part of the unstable manifold is so strong that the disappearance of L^* is smoothly followed by the birth of a stable large-amplitude periodic orbit upon crossing the stability boundary.

2. The model

Example. Consider the family of systems

$$\begin{cases} \dot{x} = x(2+\mu - B(x^2+y^2)) + z^2 + y^2 + 2y \equiv P, \\ \dot{y} = -z^3 - (1+y)(z^2+y^2+2y) - 4x + \mu y \equiv Q, \\ \dot{z} = (1+y)z^2 + x^2 - \varepsilon \equiv R, \end{cases} \tag{1}$$

where μ, ε, and B are parameters (we let $B = 10$). The system 1 for $\mu = \varepsilon = 0$ has a closed integral curve given by ($x = 0$, $z^2 + (y+1)^2 = 1$) in the phase space. There are two equilibrium states on this curve: the low one $O'(0,-2,0)$ is a saddle-node with one zero $\lambda_1 = 0$ and two negative characteristic exponents found from the equation $\lambda^2 + 40\lambda + 68 = 0$; the upper equilibrium state $O(0,0,0)$ has one zero exponent $\lambda_1 = 0$ and a pair of purely imaginary characteristic exponents $\lambda_{2,3} = \pm 2i$ (Figure 3a). Therefore, the point O' is of codimension one because $R_{zz} \neq 0$, whereas the point O is of codimension three because the two-dimensional divergence $\sigma(z) = P'_x + Q'_y = -z^2 + \cdots$ at O starts with a quadratic term [**4**]. This means that a double (semistable) cycle can be generated from O in the (x,y)-plane because "an embryo" of the first Lyapunov value vanishes here.

The unfolding of the bifurcation diagram is shown in Figure 2. It includes three bifurcations, all of codimension one.

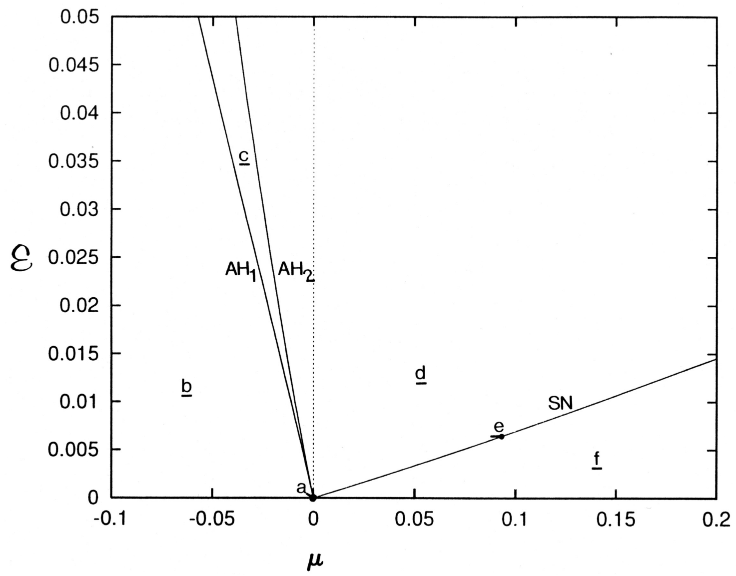

FIGURE 2. The (μ, ε)-bifurcation diagram for $B = 10$.

Let us describe the bifurcation occurring while moving in the clock-wise direction around the origin on (μ, ε)-parameter plane. When ε becomes positive, the saddle-node O' disappears, while the equilibrium state O is decomposed into two equilibria O_1 and O_2, where $z_{O_{1,2}} = \mp\sqrt{\varepsilon} + \dots$. In the region $\underline{b}$, the point O_1 is stable and O_2 is a saddle-focus of type $(2,1)$ whose one-dimensional separatrices converge to O_1 as $t \to +\infty$ (Figure 3b). Upon entering the region $\underline{c}$, the point O_1 loses its stability through a super-critical Andronov–Hopf bifurcation (this is guaranteed by the choice of large $B > 0$) on the curve AH_1 and becomes a saddle-focus (1,2). Here, both unstable separatrices of the saddle-focus O_2 tend to a new stable periodic L_1, see Figure 3c. The equilibrium state O_2 undergoes a similar Andronov–Hopf bifurcation on the curve AH_2 and becomes a totally repelling point in the region d. The unstable manifold of the just born saddle periodic orbit L_2 continues to tend to L_1, as shown in Figure 3d. On the bifurcation curve SN, both periodic orbits coalesce, thereby composing a saddle-node cycle L^* whose unstable manifold, due to continuity, is bi-asymptotic to L^* as $t \to \pm\infty$, as shown in Figure 3e. The cycle L^* vanishes in the region f, but the local stability of the system is inherited by a new, unique, stable large-amplitude periodic orbit L_{bs}, which is not homotopic to either of the former cycles, see Figure 3f.

If one attains the curve SN from the region f, the number of scrolls of the orbit L_{bs} increases uncountably near the "phantom" of the saddle-node orbit L^* in the phase space. Since both equilibrium states O_1 and O_2 are located at a finite distance from the bifurcating periodic orbit, the infinite perimeter of the latter implies its infinite length on the boundary SN, and vice versa; indeed, the flight time of any trajectory passing close by the helix-like density of orbits, from which the saddle-node limit cycle appears, tends to infinity as one gets closer to SN. Thus, we have a mechanism for the blue sky catastrophe bifurcation of the desired codimension one in the two-parameter family. It is worth noticing that the corresponding stability boundary is classified as safe in the sense that the representing phase point does not desert towards any other attractor after the bifurcation, which is apparently invertible.

The stability analysis of this system in a small neighborhood $\mu = \varepsilon = 0$ is carried out analytically. The local consideration does not involve numeric computations unless we need to show that the one-dimensional stable manifold $W^s_{O_1}$ of the saddle-focus O_1 is not enclosed in the unstable manifold of the periodic orbit L_{bs}, similarly to Figure 3d, but for parameter values on the curve SN. Otherwise, this would lead to the appearance of a two-dimensional torus instead of the stable periodic orbit. To verify this condition, one should continue $W^s_{O_1}$ backward in time in order to find its intersection point with some transverse cross-section drawn nearby on the left from O_1, and to check that this point is not surrounded by the trace of the intersection of $W^u_{L_{bs}}$ with the cross-section. This turns out to be true indeed, because our construction follows the phenomenological scenario in [**3**]; namely the presence of saddle-node point O' on the integral curve at the initial stage guarantees the further contraction in this region, so that by that time the manifold $W^u_{L_{bs}}$ passes by the hollow and reaches the cross-section, its trace finally shrinks almost to a point distant from the corresponding intersection point of $W^s_{O_1}$.

If we step back from a vicinity of the origin ($\mu = \varepsilon = 0$), the unstable manifold $W^u_{O_2}$ of the saddle-focus O_2 may no longer follow along the prescribed global pathway. Of special interest is the situation where $W^u_{O_2}$ becomes homoclinic to O_2. Provided that Shilnikov's condition holds here, i.e., the saddle value is positive at the saddle-focus near such a homoclinic loop, one should expect the onset of dynamical chaos, as depicted in Figure 4.

Moreover, we should emphasize that the bifurcation of an equilibrium state with eigenvalues $(0, \pm i\omega)$ may, under some matching condition, generate "miniature" chaos induced by heteroclinic connections between both saddle-foci, as well as by homoclinic loops to either one. The associated chaotic limit set may be a strange attractor or a strange repeller (an attractor as $t \to -\infty$), depending locally on

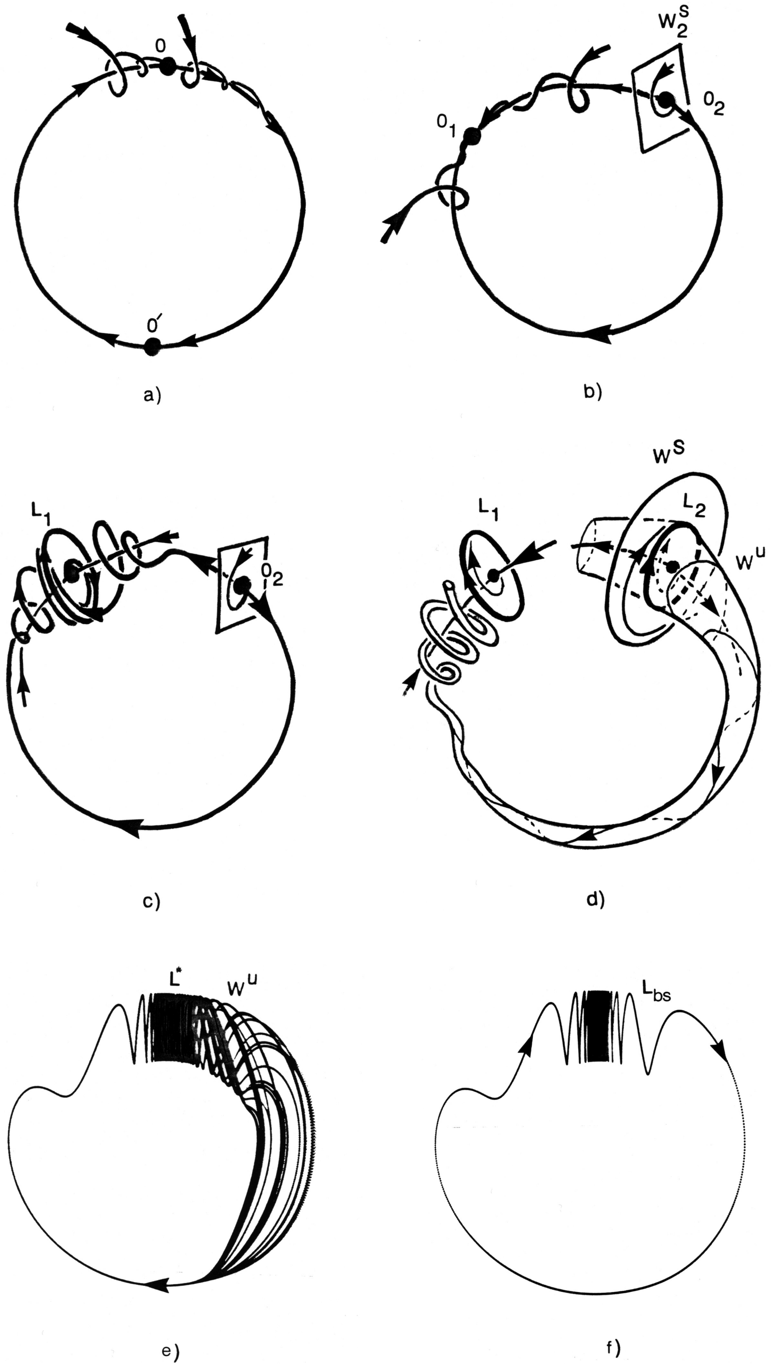

FIGURE 3. Mechanizm of the blue-sky catastrophe. (f) A stable periodic orbit at ($\mu = 0.3, \varepsilon = 0.021$)

the sign of divergence of the vector field. There is a 3D-exotic example of this bifurcation resulting in the co-appearance of the chaotic spiral repeller and attractor

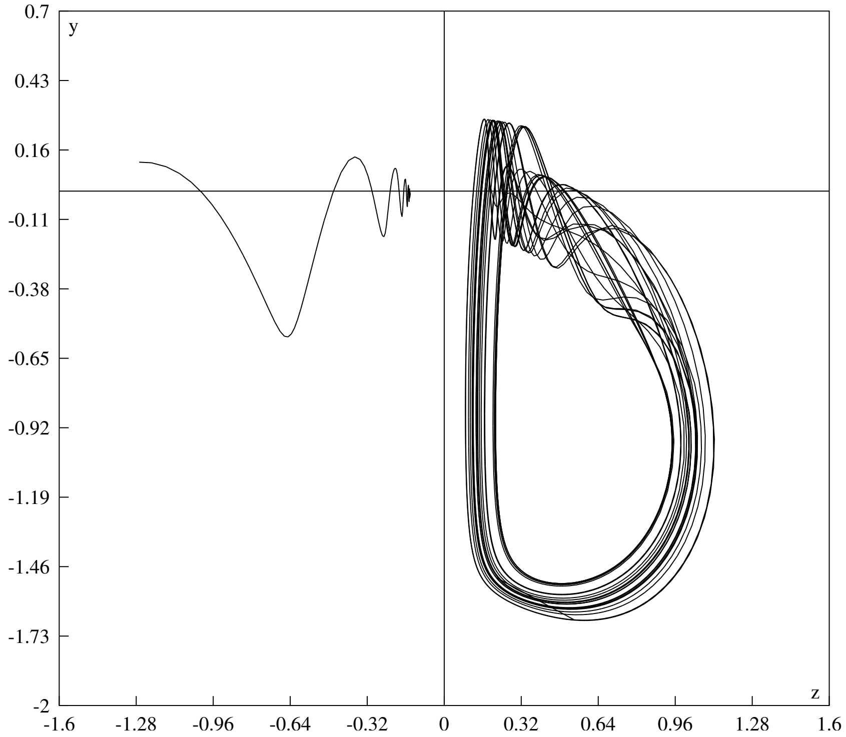

FIGURE 4. Spiral chaos in region b for ($\mu = -0.173$, $\varepsilon = 0.01$).

which are the α- and ω-limit sets, respectively, for trajectories close in the phase space to such a twice-degenerate equilibrium state [**5**].

3. Conclusion

To conclude, we note that the blue-sky catastrophe in action may be helpful for a qualitative explanation of the frequently observed transition from low-amplitude oscillations (spikes) to large-amplitude burstings in models for neuron activity, or to flow surges in models of jet engines.

References

[1] L. P. Shilnikov and D. V. Turaev, *On a blue sky catastrophe*, Dokl. Akad. Nauk SSSR **342** (1995), no. 5, 596–599; English transl. in Russian Math. Dokl.
[2] ———, *Simple bifurcations leading to hyperbolic attractors*, Computers Math. Applic. **34** (1997), no. 2–4, 173–180.
[3] ———, *A new simple bifurcation of a periodic orbit of "blue sky catastrope" type.* This volume.
[4] N. K. Gavrilov, *On some bifurcations of an equilibrium state with one zero and a pair of purely imaginary roots*, Methods of Qualitative Theory of Differential Equations, Gorky State University, Gorky, 1978.
[5] A. L. Shilnikov, C. Nicolis and G. Nicolis, *Bifurcation and predicability analysis of a low-order atmospheric circulation model*, Int. J. Bifurcation and Chaos **5** (1995), no. 6, 1701-1711.

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