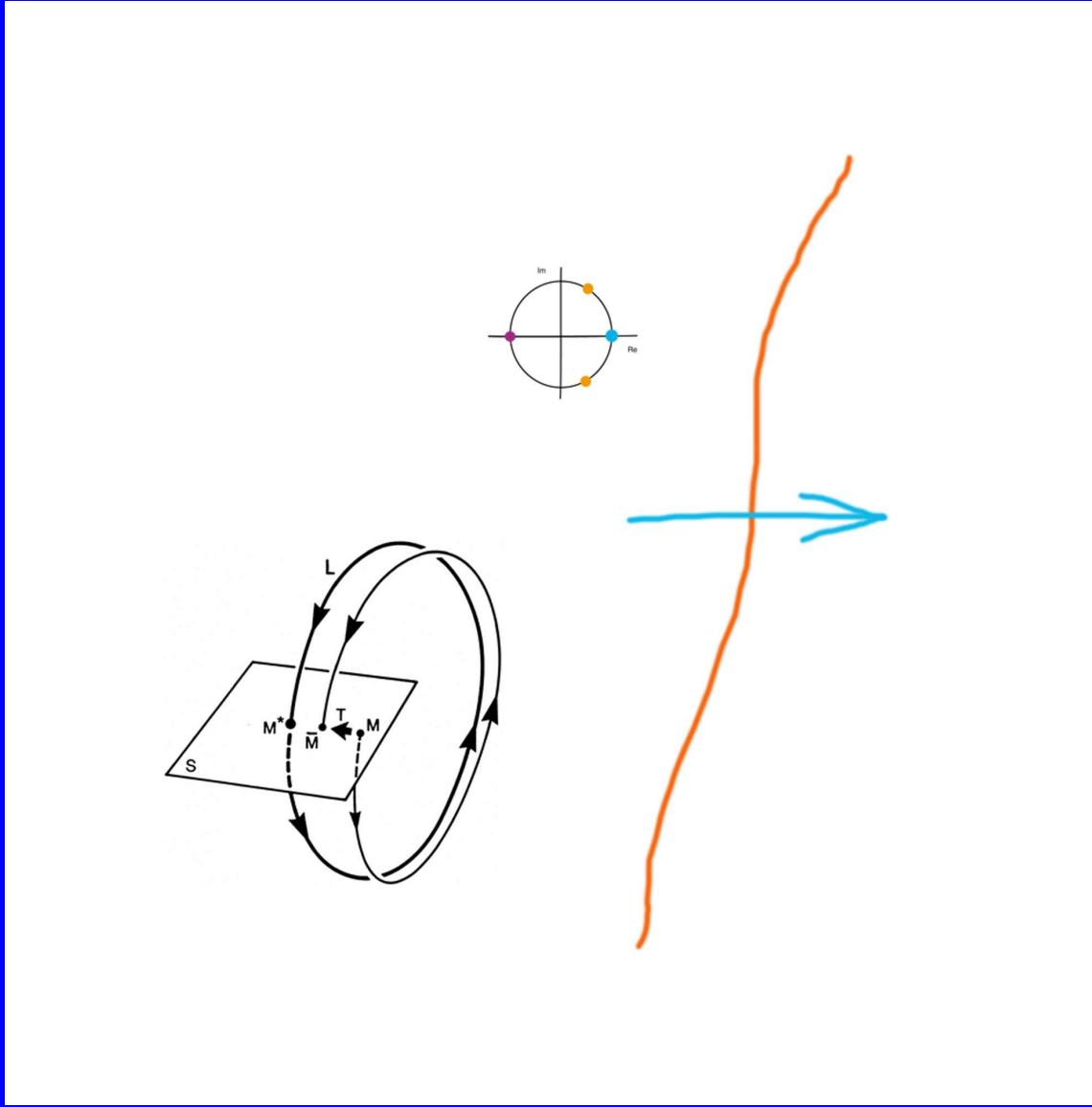


Infinite cycles in slow-fast systems

“Blue Sky Catastrophe”

Andrey Shilnikov, GSU, Atlanta

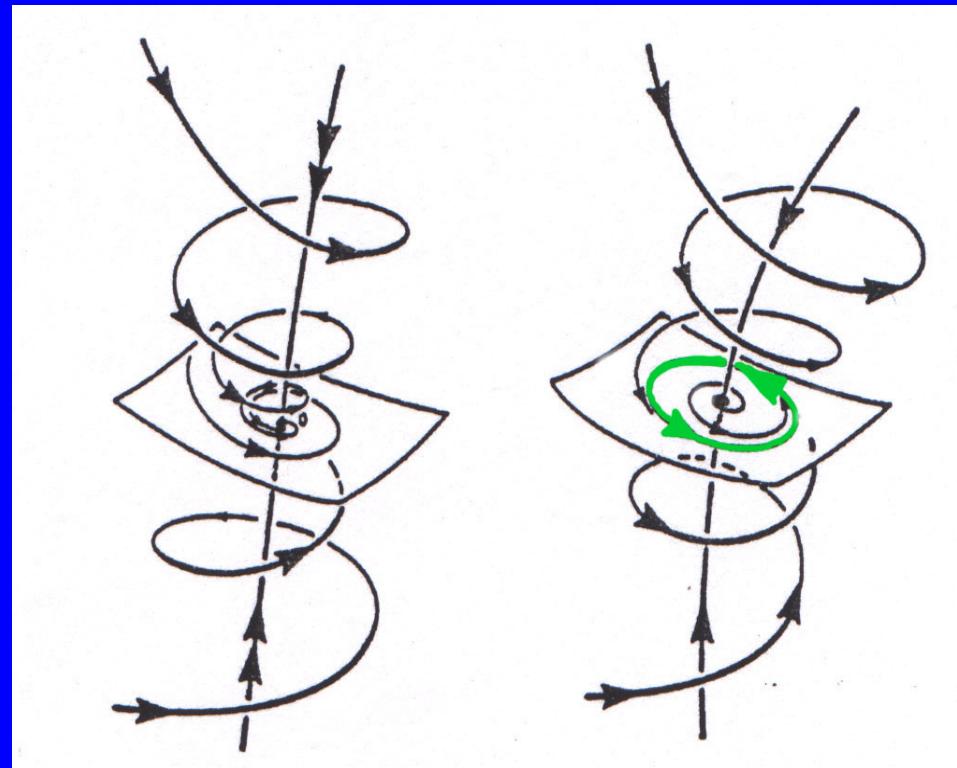


T and *L* classification

Periodic orbit on stability boundary

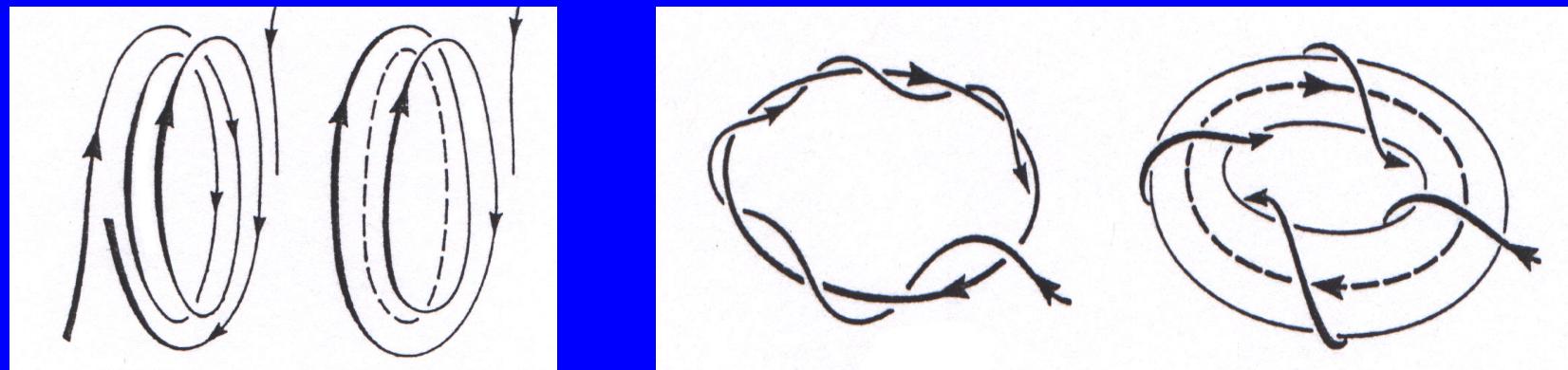
- Group I : finite period T , length $L \rightarrow 0$
- Group II : finite period T and length L
- Group III: period $T \rightarrow \infty$, length L finite
- Group IV: $T \rightarrow \infty$ and $L \rightarrow \infty$???

Group I : finite period T , length $L \rightarrow 0$



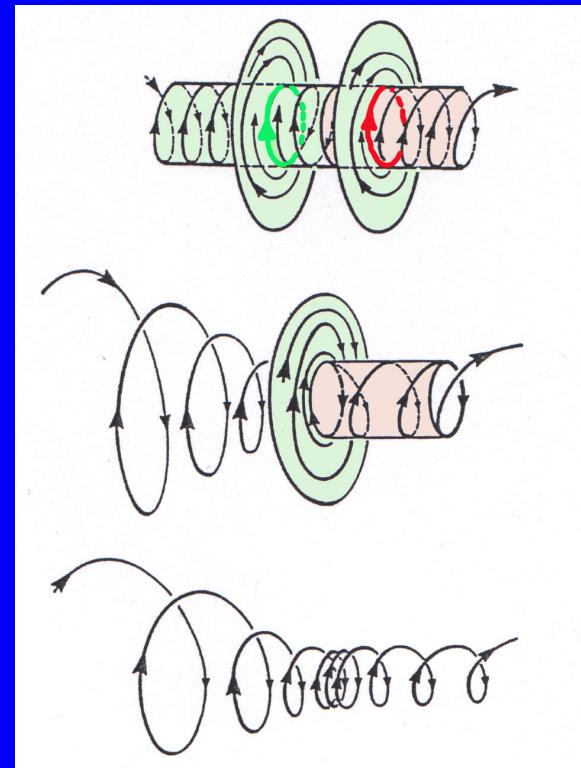
Supercritical - Hopf bifurcation

Group II : both T and L finite



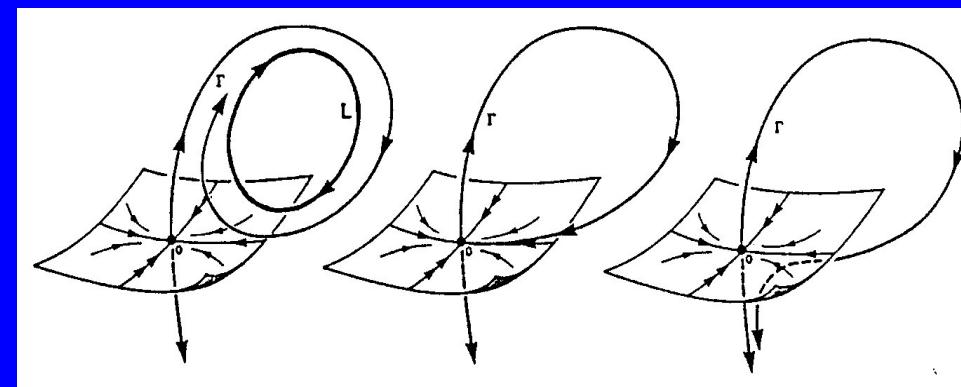
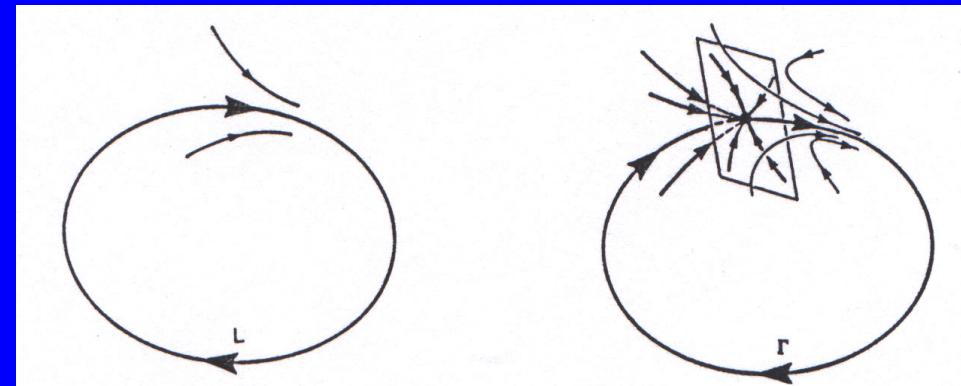
Period doubling, $\mu = -1$; Torus bifurcation $\mu_{1,2} = e^{\pm i\omega}$

Group II : both T and L finite



Plain saddle-node cycle , $\mu = +1$

Group III : $T \rightarrow \infty$, finite L



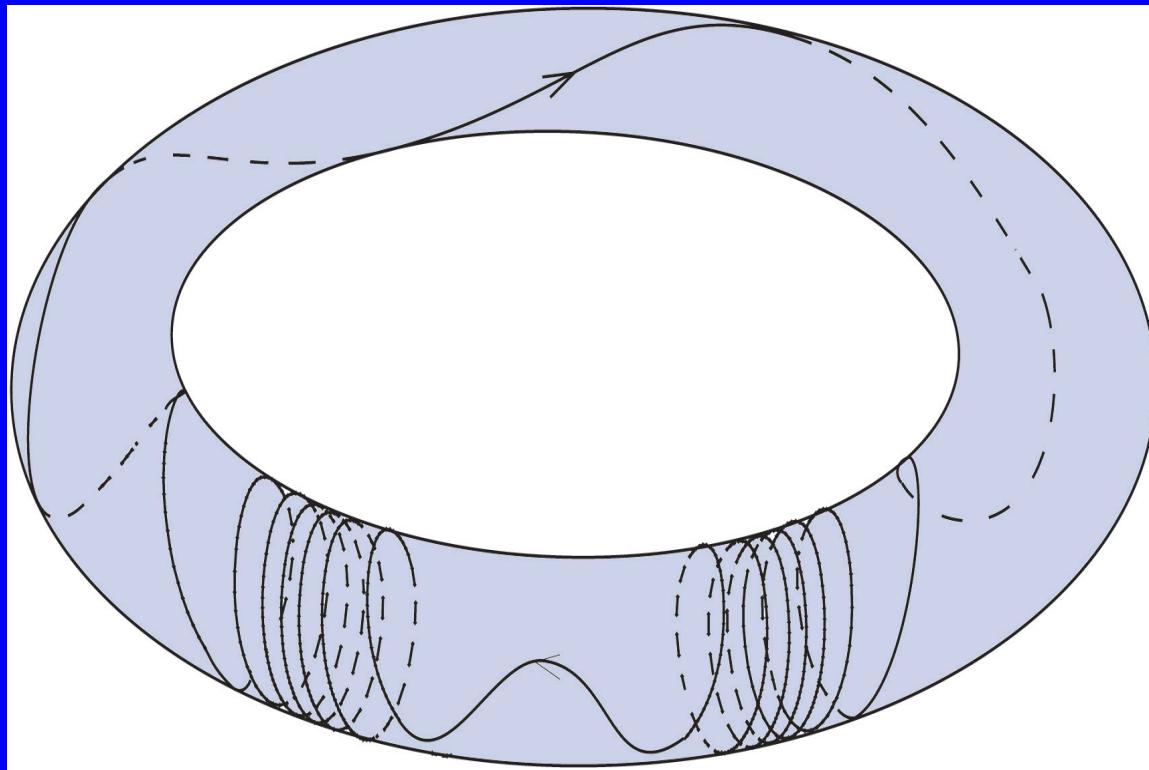
Homoclinic saddle-node and saddle

Palis-Pugh problem

Limit cycle with $T \rightarrow \infty$ and $L \rightarrow \infty$, in addition

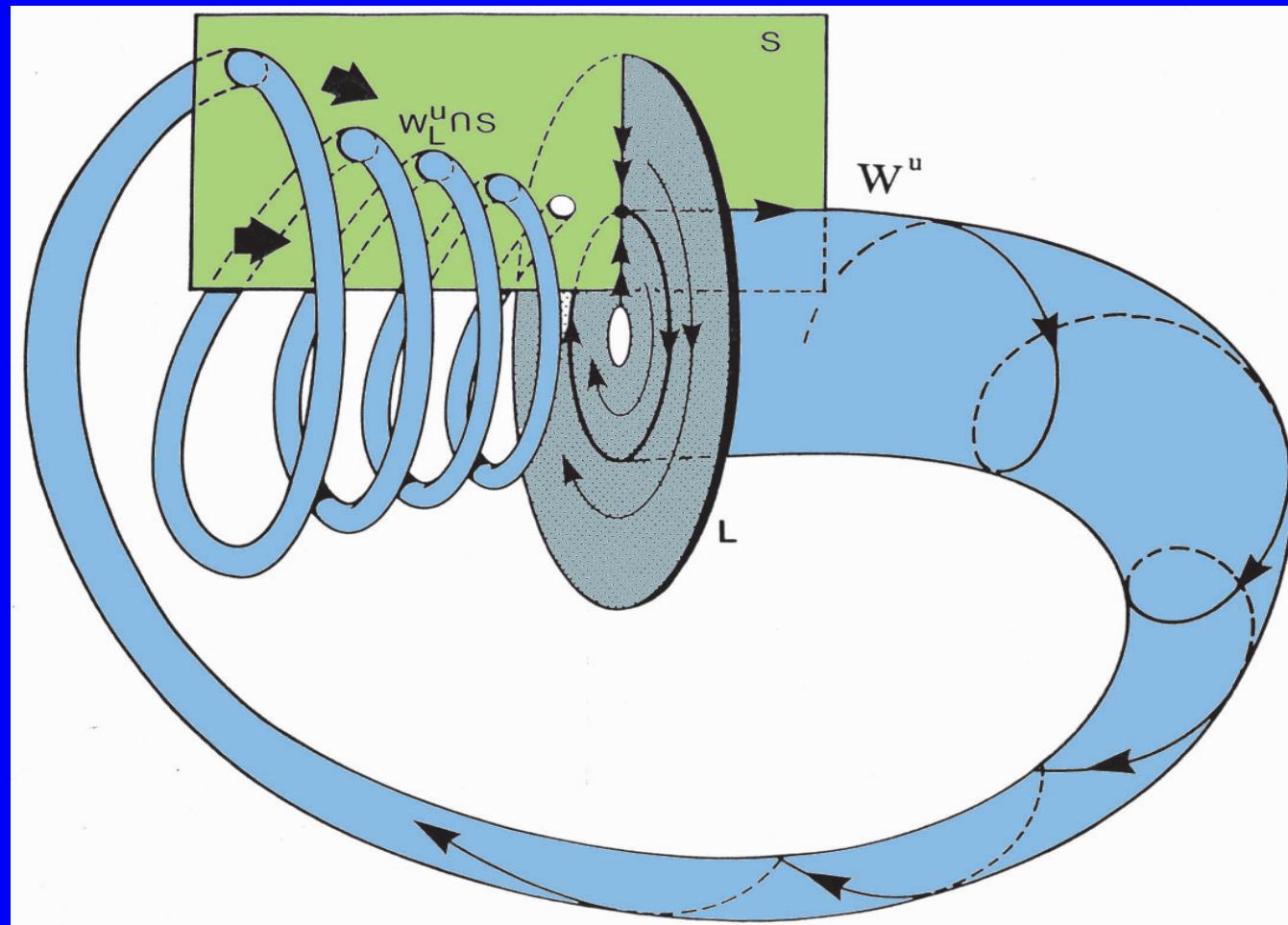
1. of Codimension 1;
2. no bifurcations on the route to the stability boundary
3. no equilibria involved (where $\dot{X} = 0$)

Hypothetic example I

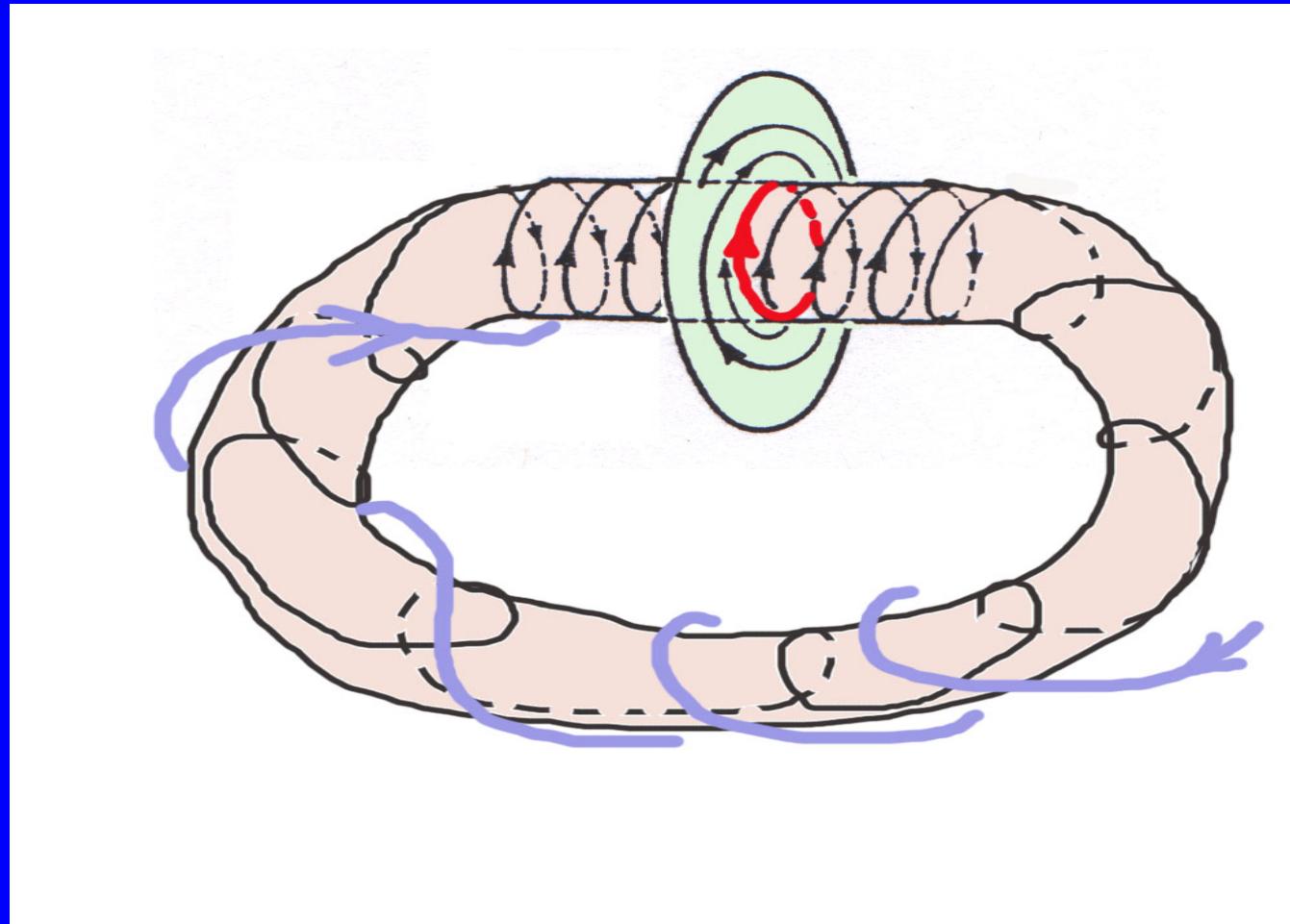


Special resonant torus

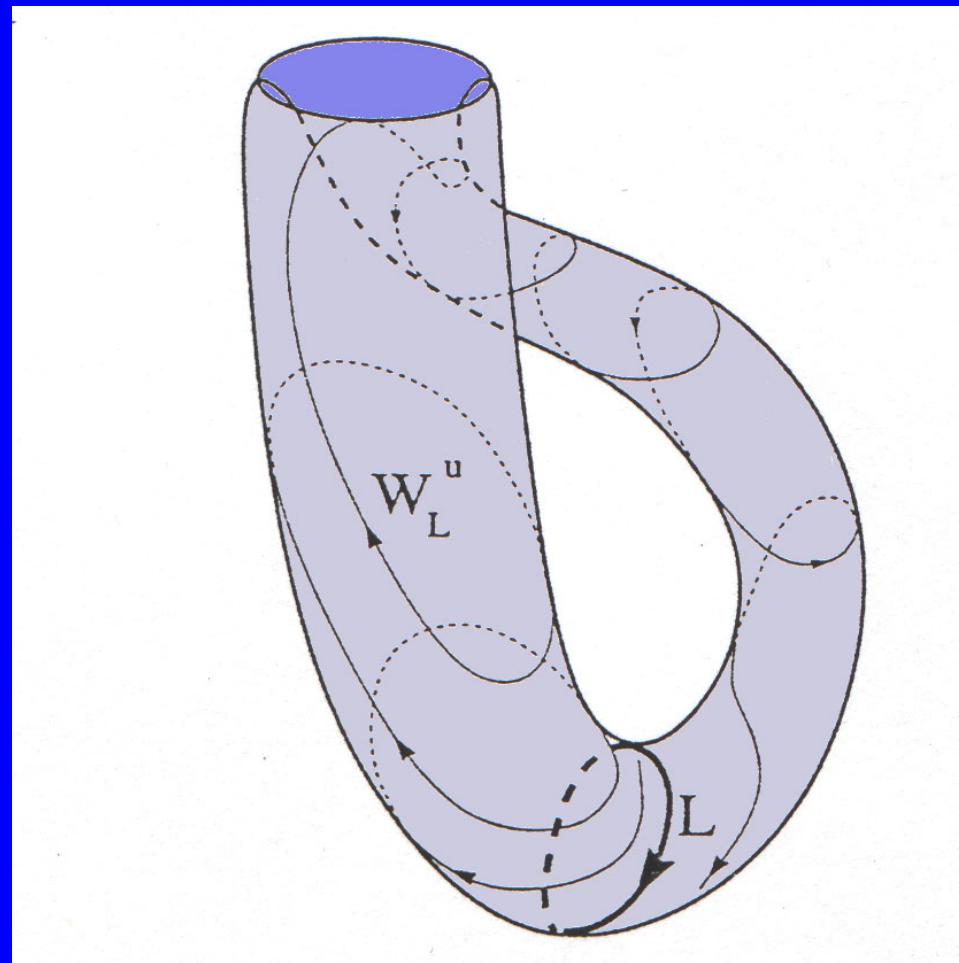
Shilnikov-Turaev construction



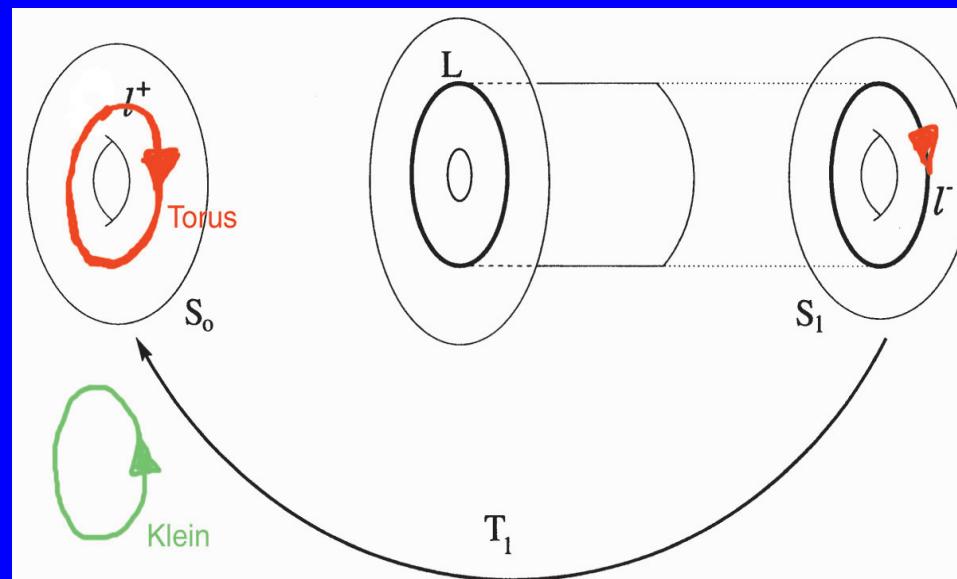
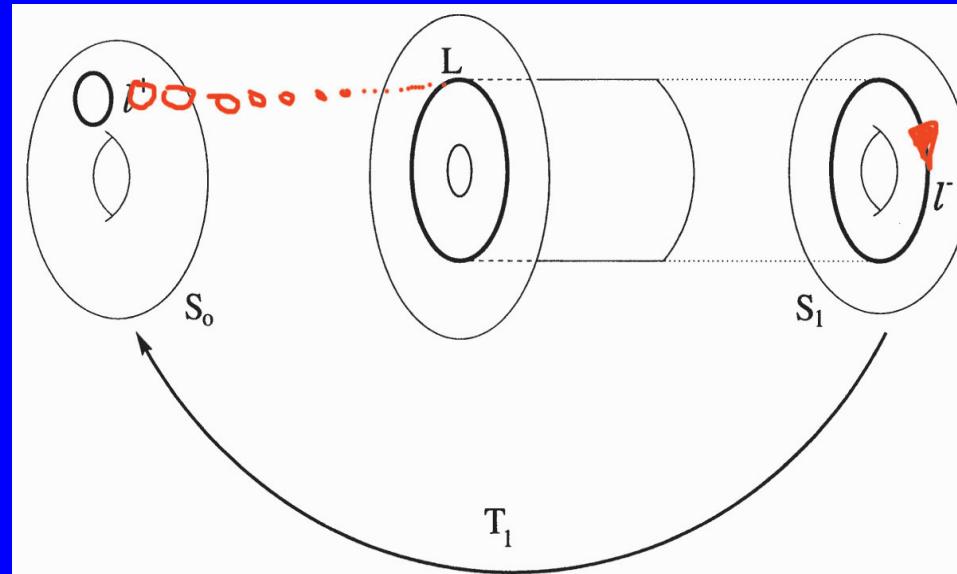
Blue sky bifurcation



Homoclinic Saddle-node cycle \Rightarrow Resonant torus



Klein bottle



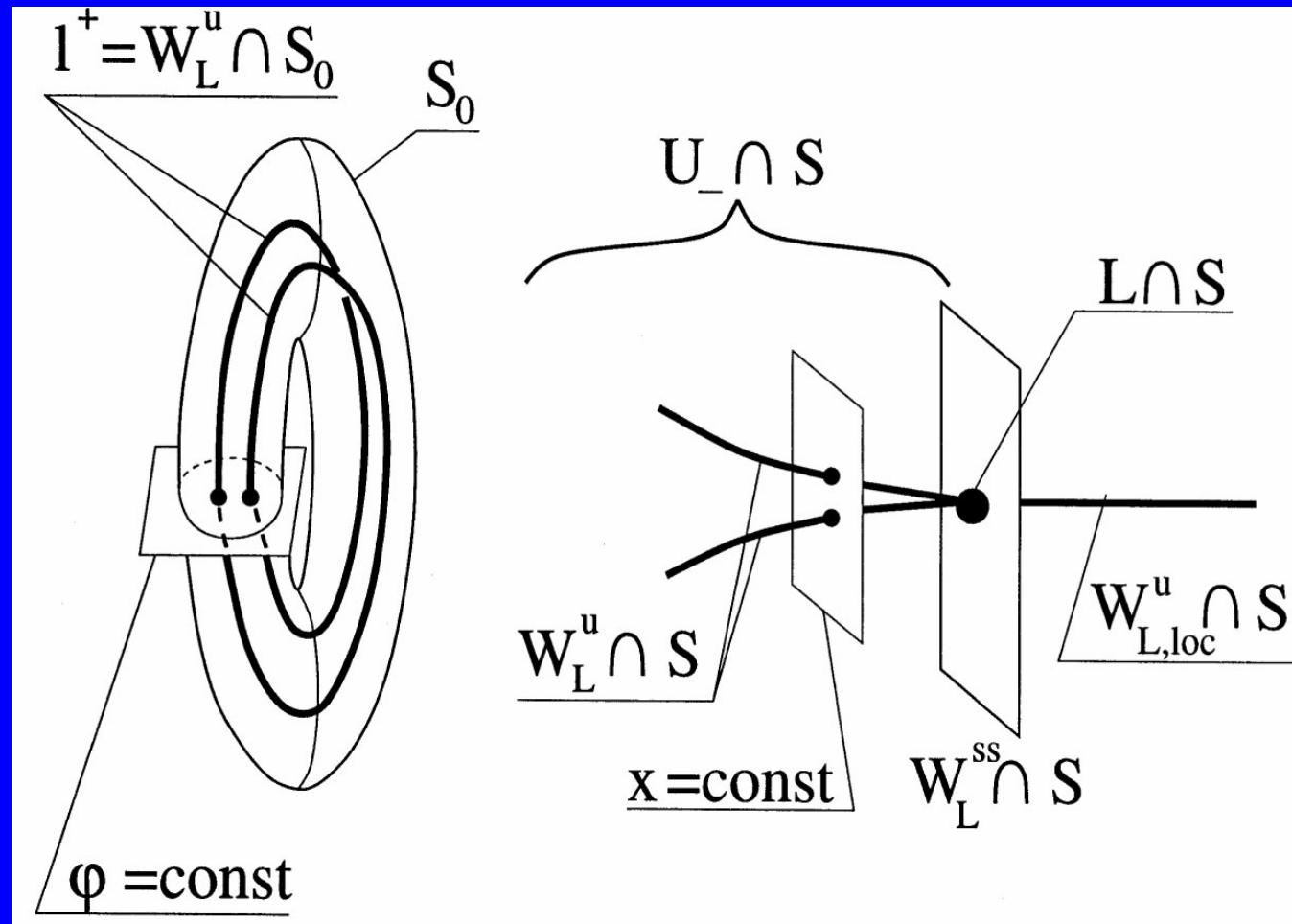
Non-manifold case, Torus (orientation preserved) and Kl  n bottle

Essential mapping on S_1

$$\begin{aligned}\bar{y} &= 0, \\ \bar{\varphi} &= \omega(\mu) + m\varphi + f(\varphi, \mu), \quad \text{mod} \quad 1\end{aligned}$$

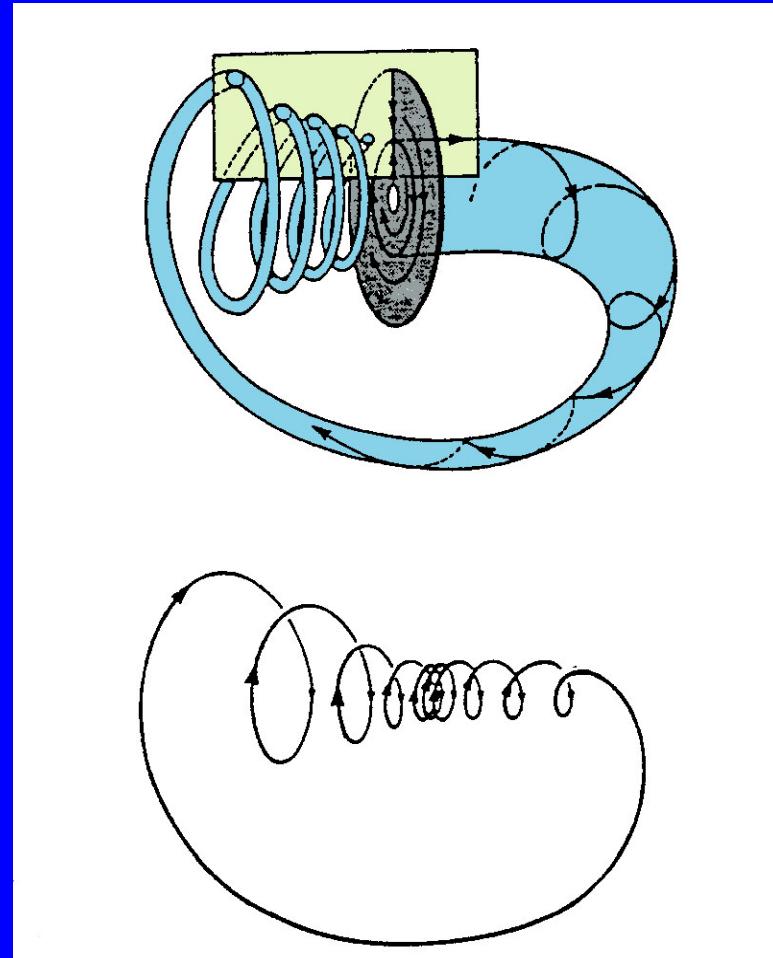
where

- Nonmanifold $m = 0$
- Torus +1
- Klein bottle -1 in R^4
- (next page)



William's solenoid $m = 2$

In case of $m = 0$, let $|f_\varphi(\cdot, 0)| < 1$.



Blue sky catastrophe in action

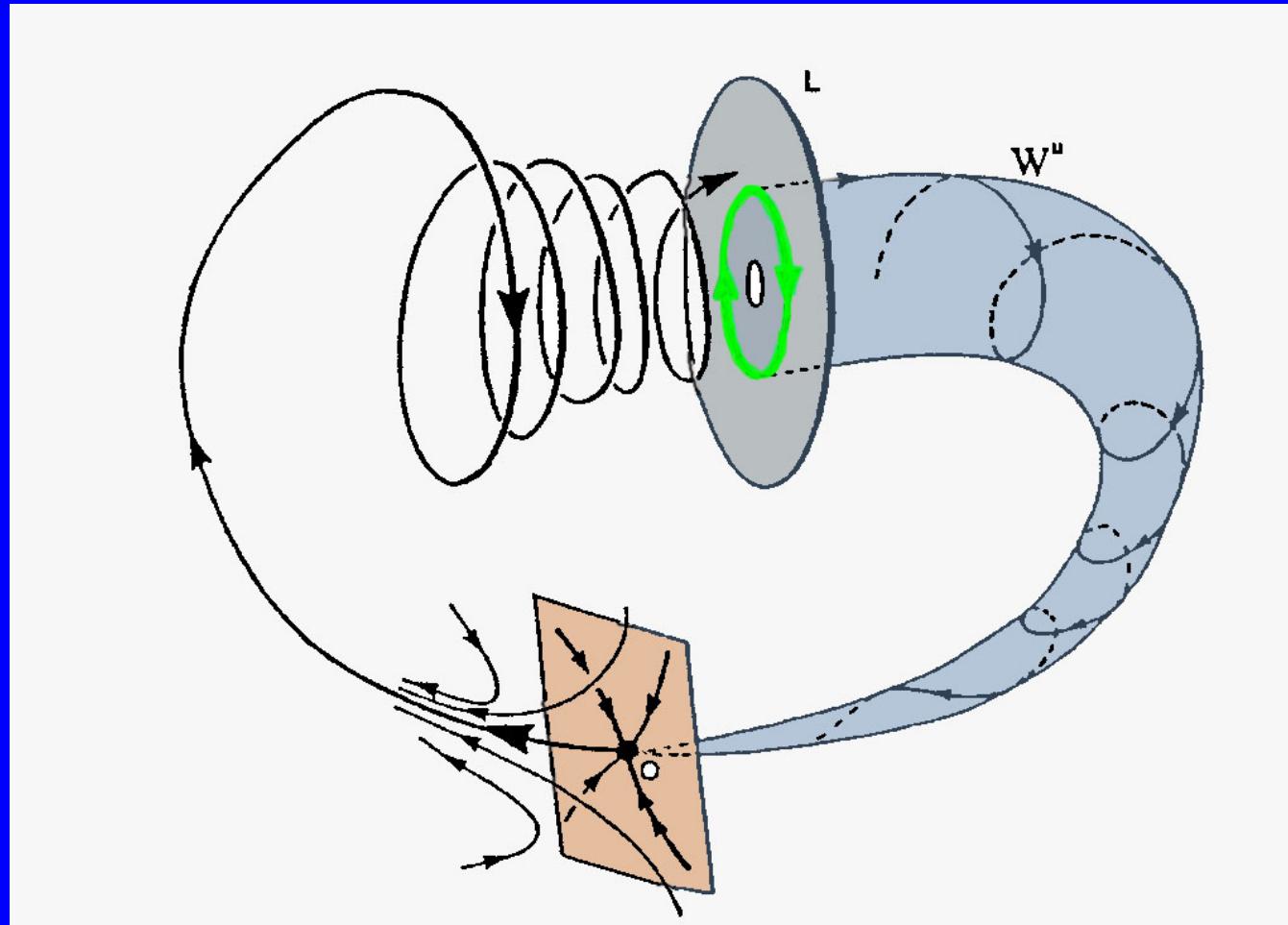
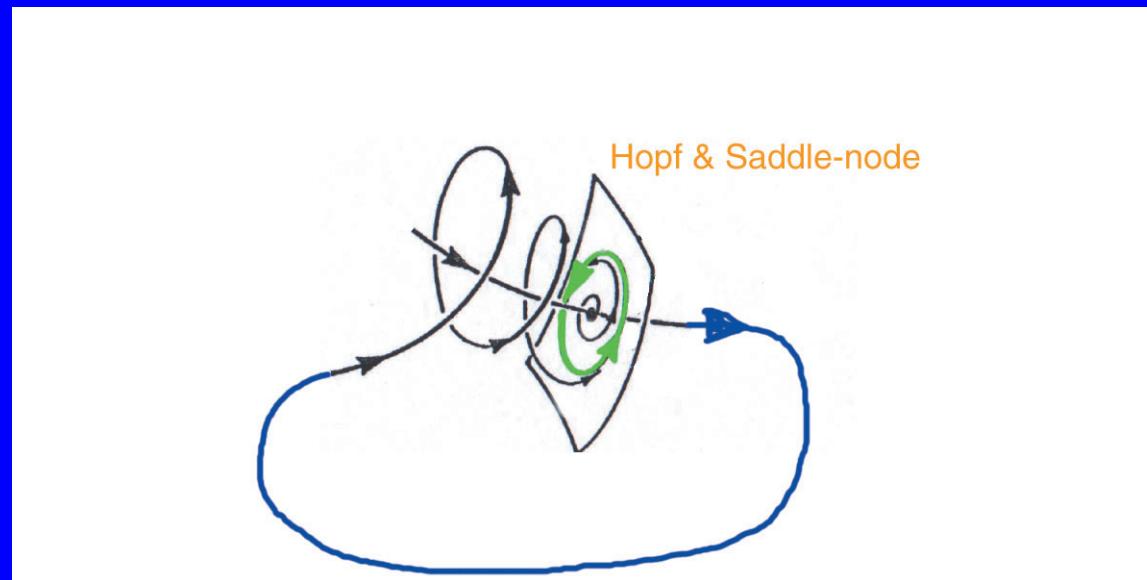


Figure 1: Contraction and squeezing

Global Hopf-Saddle-node ($\pm i\omega, 0$)

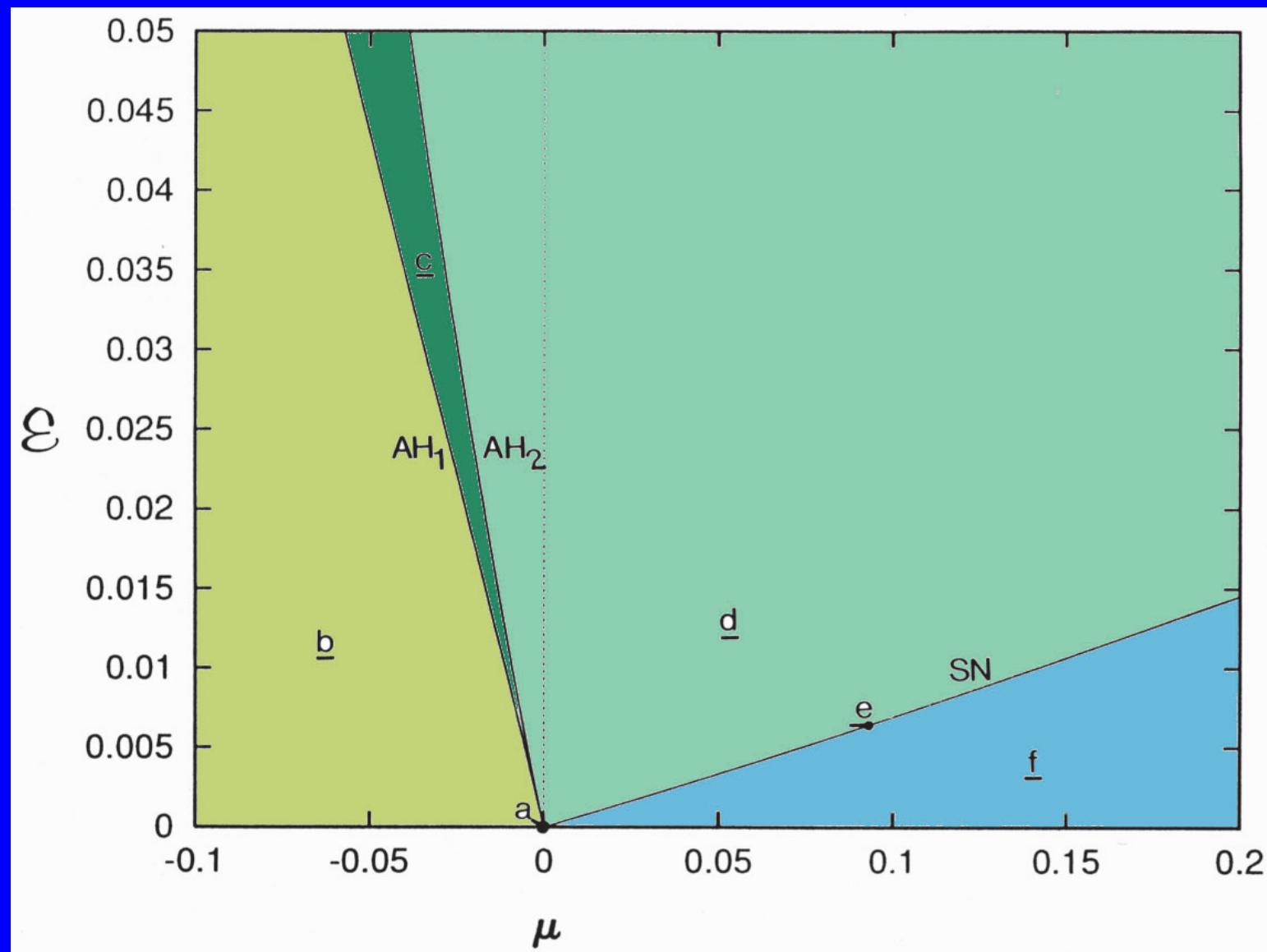
$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\dot{z} = \mu + z^2 + \dots$$



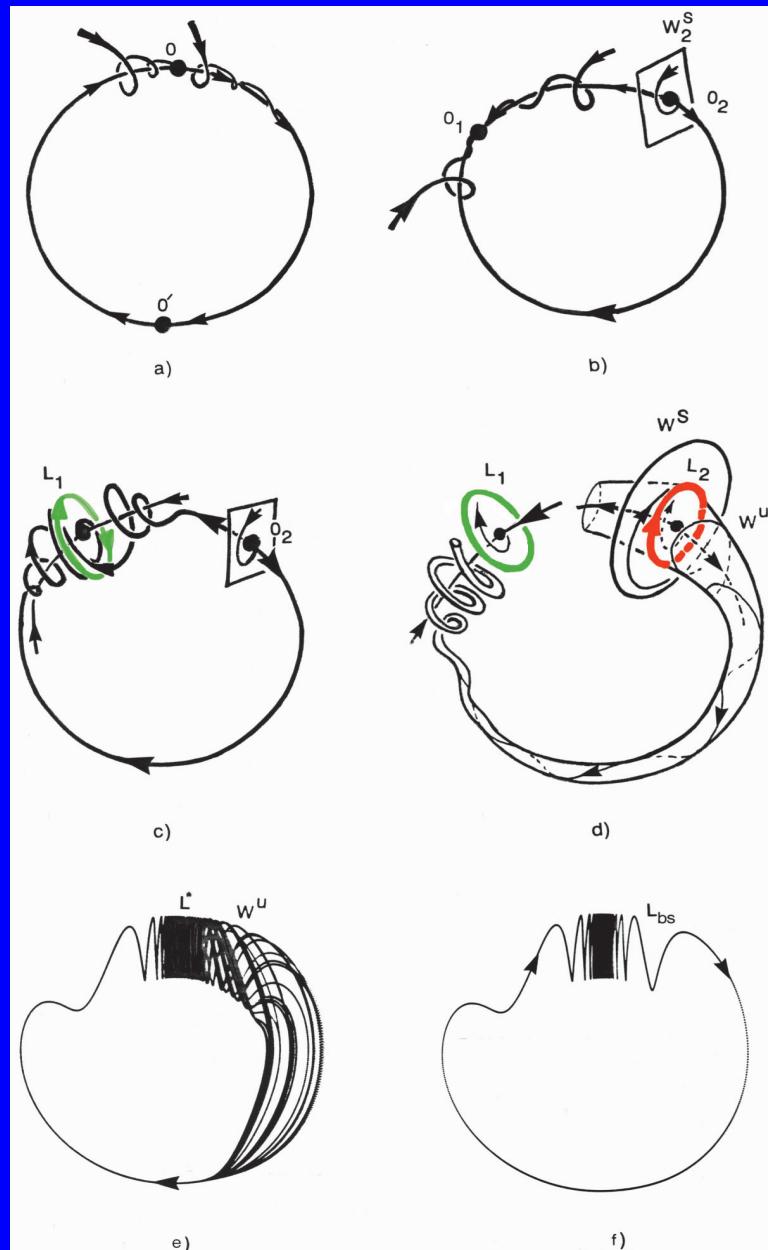
Gavrilov-Shilnikov normal form

$$\begin{cases} \dot{x} = x(2 + \mu - B(x^2 + y^2)) + z^2 + y^2 + 2y, \\ \dot{y} = -z^3 - (1 + y)(z^2 + y^2 + 2y) - 4x + \mu y, \\ \dot{z} = (1 + y)z^2 + x^2 - \varepsilon, \end{cases} \quad (1)$$

where μ , ε , and B ($= 10$) are parameters.

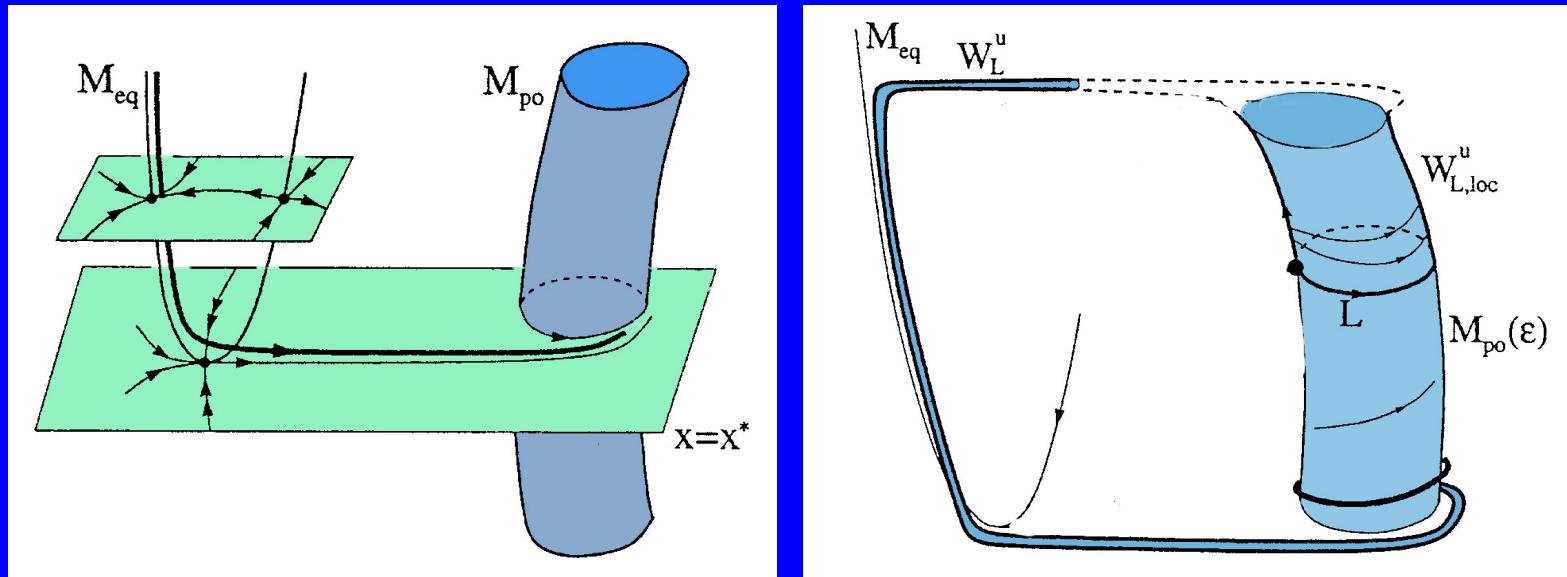


Bifurcation diagram



Phase portraits

Two time scale systems



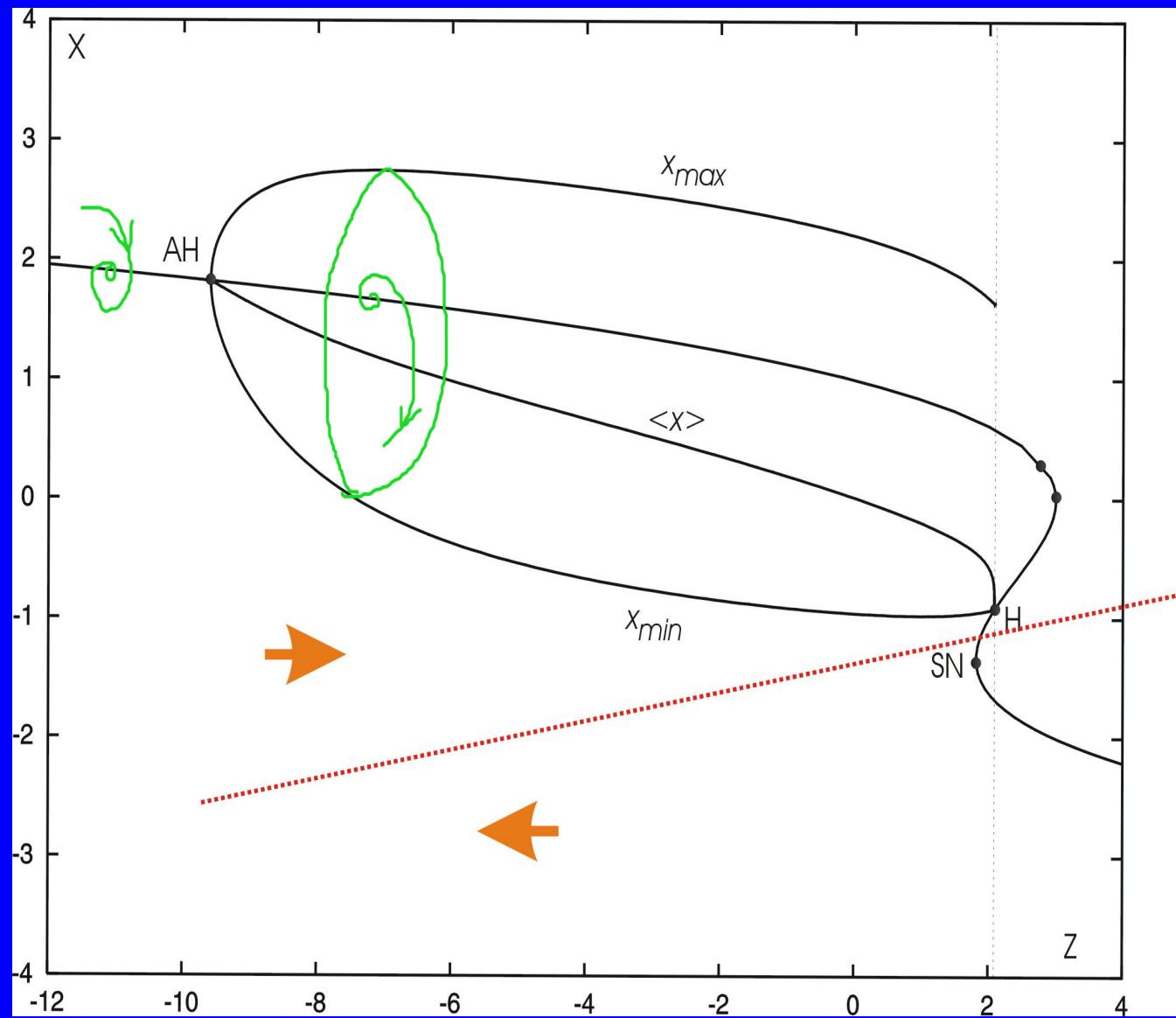
Hindmarsh-Rose model of neuron activity

$$\begin{cases} \dot{x} = y - z - x^3 + 3x^2 + 5, \\ \dot{y} = -y - 2 - 5x^2, \\ \dot{z} = \varepsilon(2(x + 2.1) - z). \end{cases} \quad (2)$$

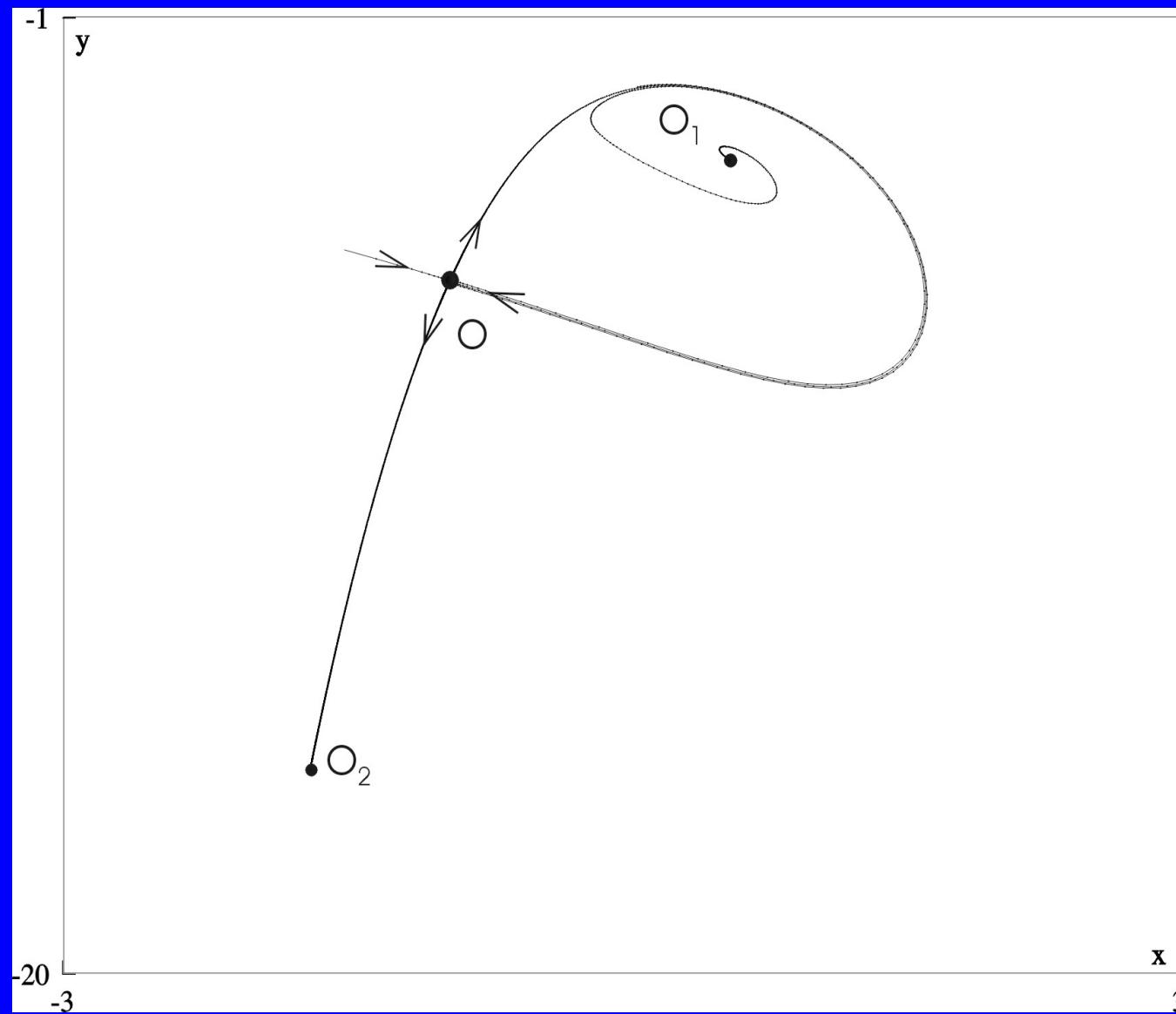
Slow sub-system

$$\begin{cases} \dot{x} = y - z - x^3 + 3x^2 + 5, \\ \dot{y} = -y - 2 - 5x^2, \end{cases} \quad (3)$$

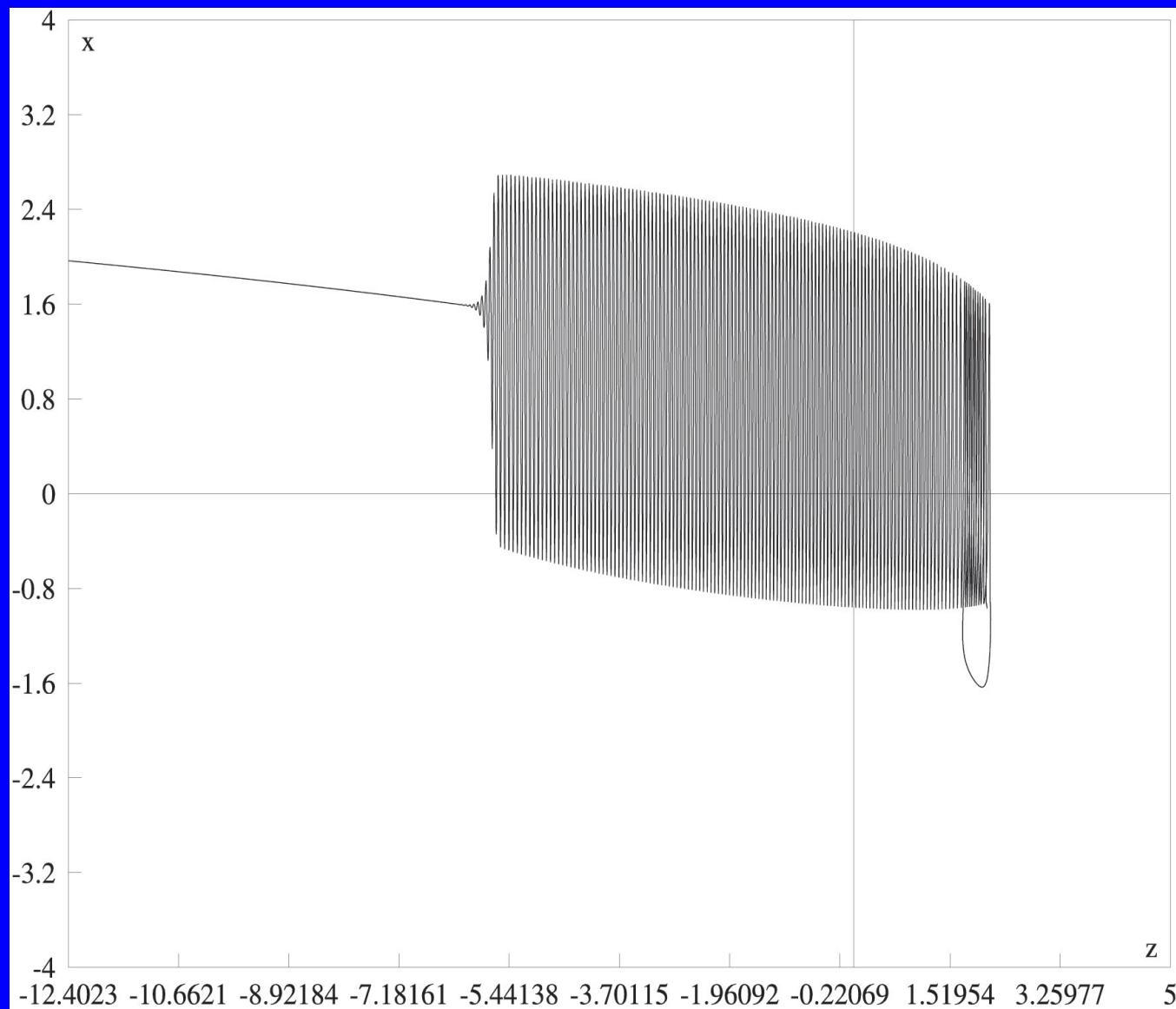
where z is a parameter.



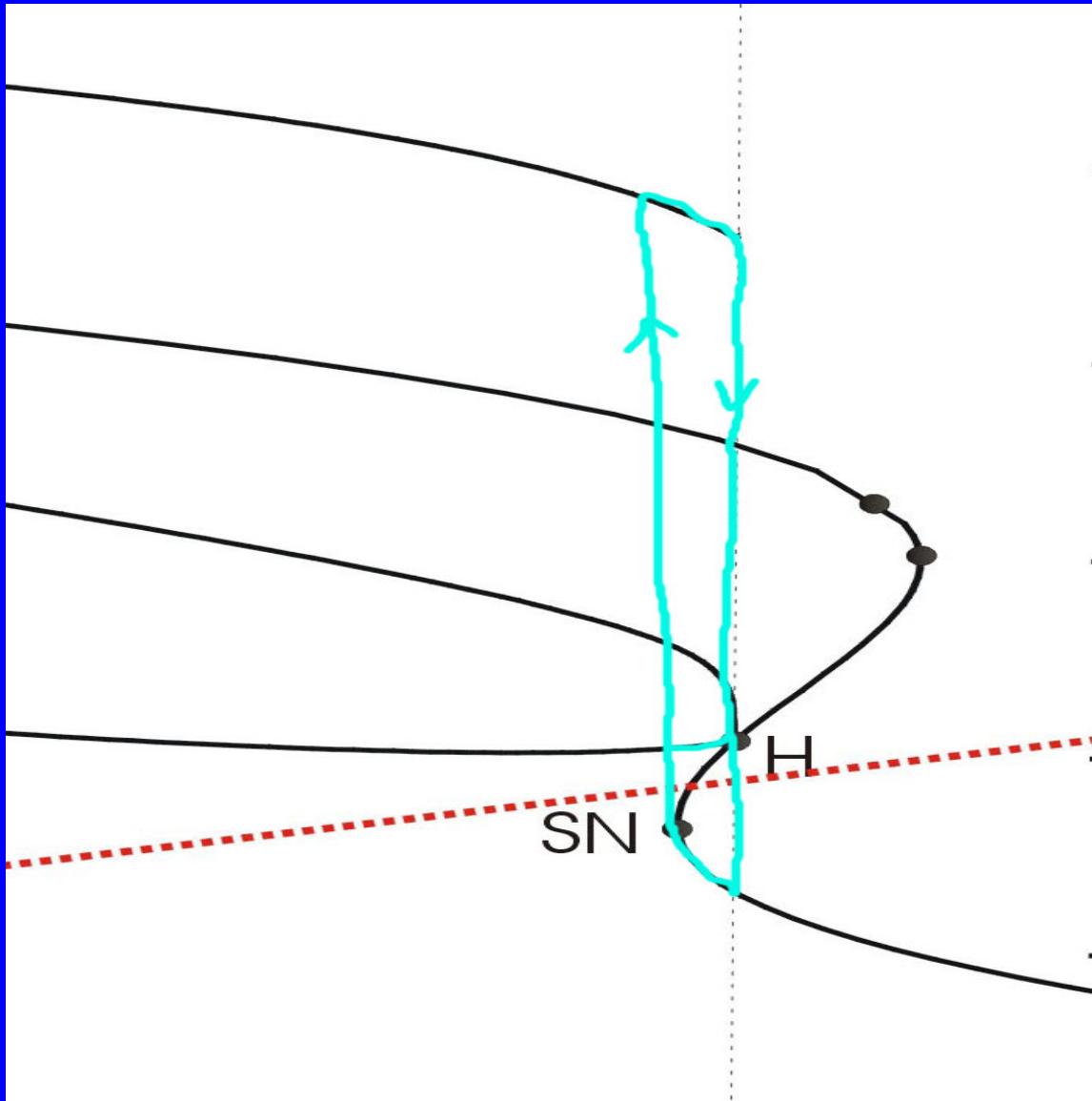
Bifurcation diagram for slow sub-system



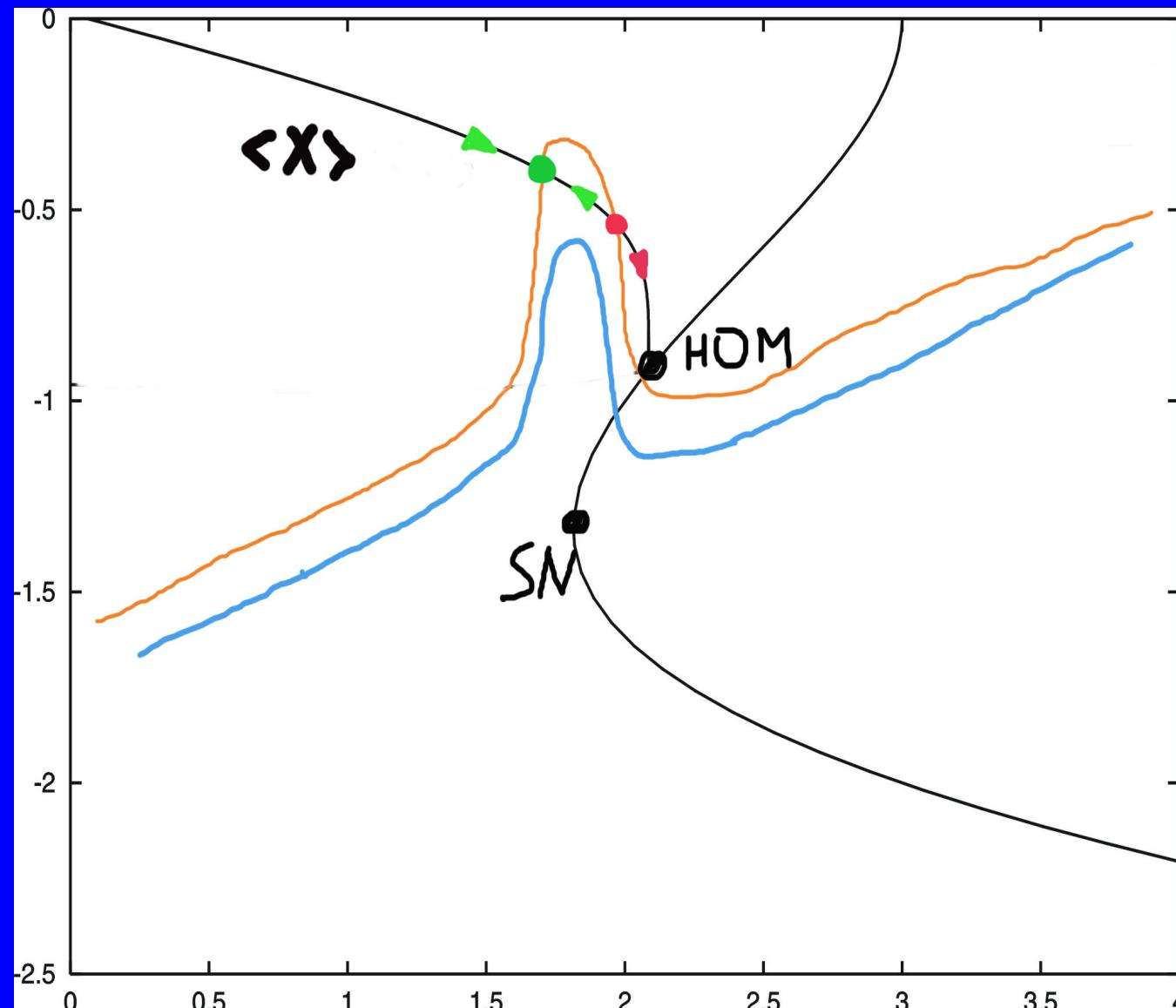
small Homoclinics in slow system



x vs. z in Hindmarsh-Rose model



Bifurcation diagram for slow sub-system



Fragment of

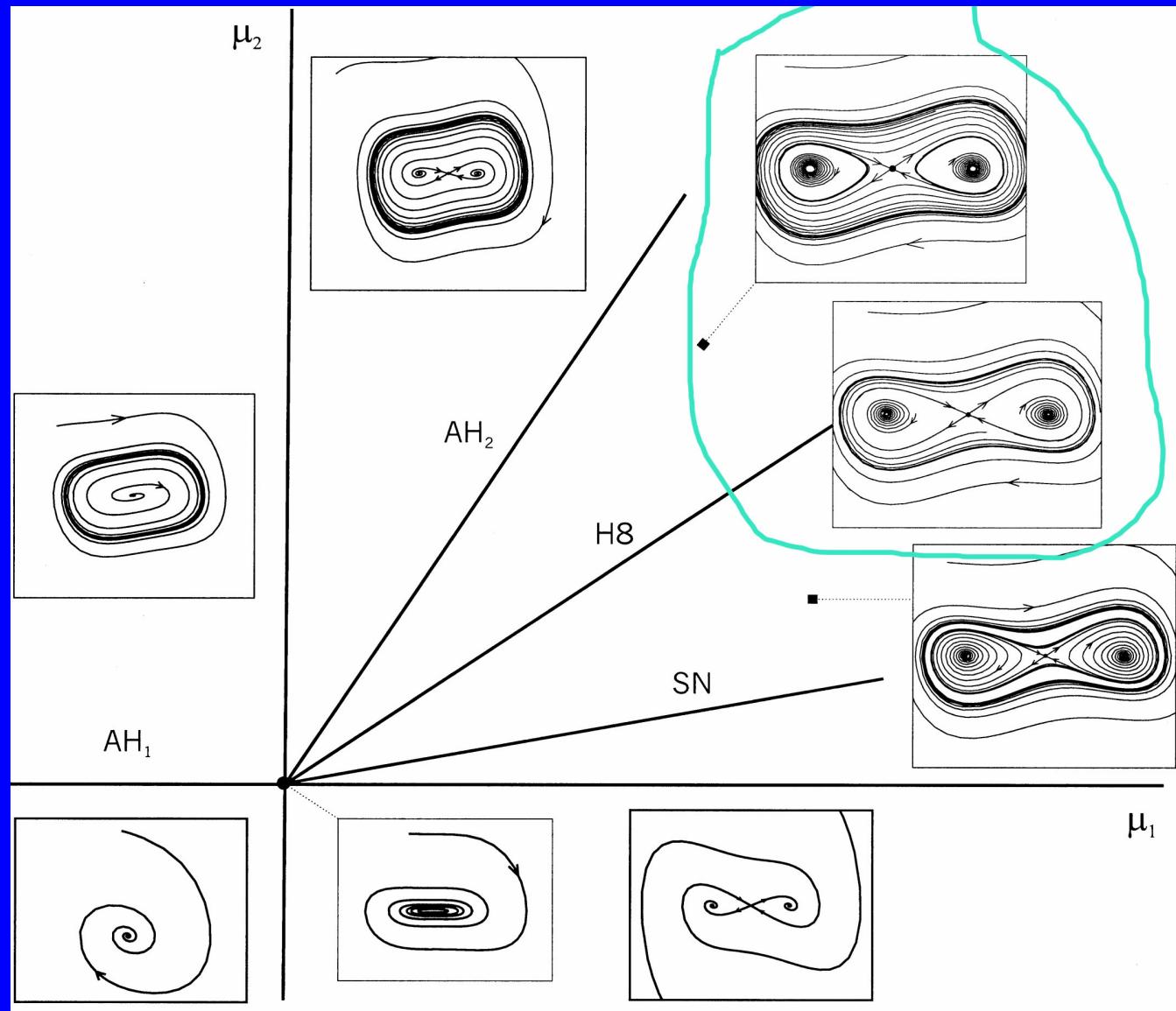
bifurcation diagram

Tuned Hindmarsh-Rose model

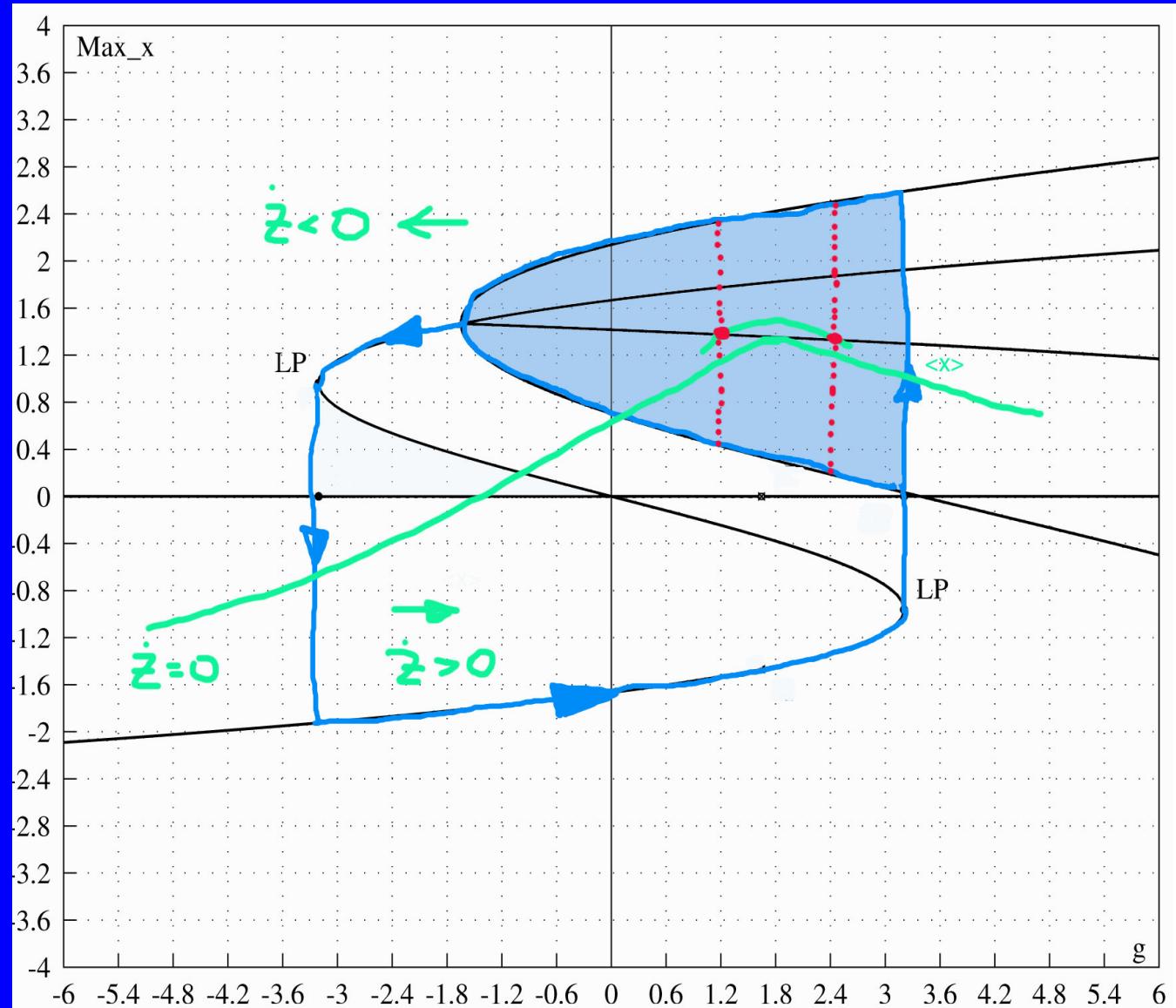
$$\begin{cases} \dot{x} = y - z - x^3 + 3x^2 + 5, \\ \dot{y} = -y - 2 - 5x^2, \\ \dot{z} = \varepsilon(2(x + 2.1) - z - \frac{A}{(z - 1.93)^2 + 0.003}). \end{cases} \quad (4)$$

Augmented Van der Pol – Duffing or Khorozov-Takens Normal form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - x^3 - \alpha y - x^2 y + z, \\ \dot{z} = \varepsilon(-x + cx(5 - x^2)). \end{cases} \quad (5)$$



Bifurcation diagram



$(x - z)$ -projection