DANGEROUS AND SAFE STABILITY BOUNDARIES OF EQUILIBRIA AND PERIODIC ORBITS

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Abstract

This tutorial reviews the state-of-the-art in bifurcation theory for smooth continuous-time dynamical systems.

1 Introduction

The study of steady (not intermittent) regimes in any applicable or phenomenological model admitting description by differential equations is known to be one of the principal elements in nonlinear dynamics theory. That may be a stationary regime, self-oscillating and periodically forced oscillations, the regime of beatings (modulation) and stochastic behaviour; in the last case, one talks about dynamical chaos.

The respective mathematical images of enumerated regimes are an equilibrium state, a periodic orbit, an invariant torus with a quasi-periodic trajectory on it (as well as multi-frequency solutions) and a strange attractor. In order that these objects might be observable they must obviously be attractive and structurally stable (insensitive to small perturbations). We notice also that the second property has relation to the two former ones only.

1.1 Stationary solutions

A stationary regime is the simplest one. The fundamental results on stability of steady states date from the last century and were mostly stimulated by the problem of auto-control of beam engines. Maxwell (for 3D systems), Routh and Hurwitz were the first who determined stability conditions for linearized equilibria in an algorithmic and easily checked form.

1.2 Periodic oscillations

The second type of established regime is periodic oscillations. For a long time, the Cartesian mechanics problem of three bodies has been the only inspiring factor in studying periodic orbits. However, breakthroughs in radio-engineering in the nineteen-twenties and thirties biased the direction towards the study of essentially nonlinear and non-conservative problems. It appeared that autonomous systems, i.e. systems having no periodic source, might exhibit a stable periodic regime. To this regime A. Andronov has given the name *self-oscillations* with the associated mathematical image of a Poincaré limit cycle. The phase portrait along with limit cycles began then to be used very widely in nonlinear planar systems at first and later in higher-order dynamical systems.

As far as periodically forced nonlinear systems are concerned, there are two different approaches which may be followed: one can either transform such a system to an autonomous one by introducing a new cyclic variable, or reduce it to a diffeomorphism over a period of an external force and then use a whole toolkit of return maps. Such original systems have been of a low order [2-4] and usually quasi-linear, thereby allowing one to apply an averaging method, i.e. to truncate the problem to a system of two equations. By that time the famous book "Theory of oscillations" by Andronov and Khaikin (the third co-author Witt appeared in the second edition of the book) came out and became a reference text for many researchers in a variety of scientific disciplines.

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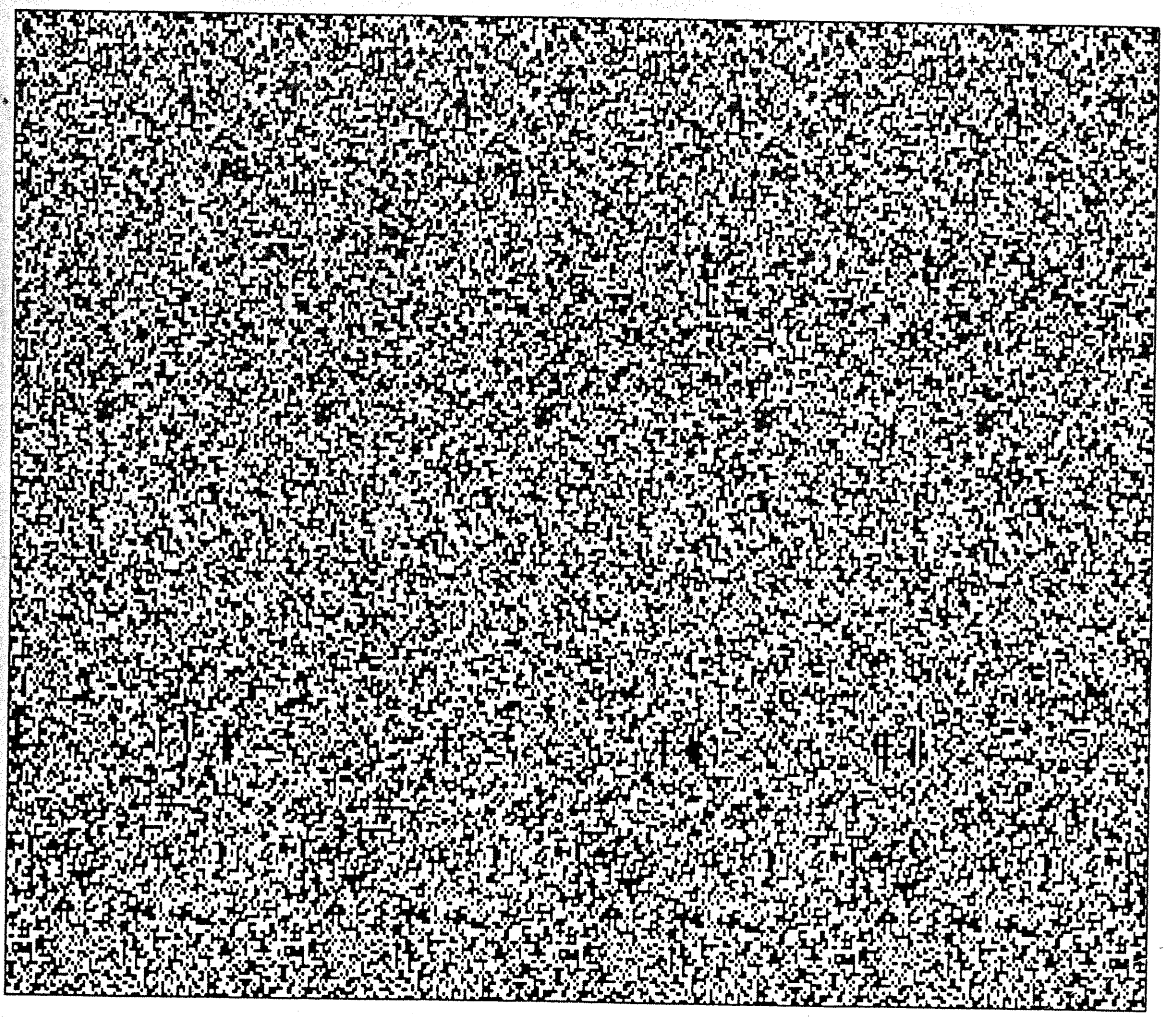

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Homoclinic codimension-2 points and 3-level laser models

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ABSTRACT

A description of the principal bifurcations which lead to the appearance of the Lorenz attractor is given for the 3D normal form for codimension-3 bifurcations of equilibria and periodic orbits in systems with symmetry. We pay special attention to two bifurcation points corresponding to the formation of a homoclinic butterfly of a saddle with unit saddle index and to a homoclinic butterfly with zero separatrix value.

1. INTRODUCTION

The Lorenz model

$$\begin{aligned} \dot{x} &= -\sigma(x-y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy \end{aligned} \tag{1}$$

is well known to be isomorphic with the equations of homogeneously broadened single-mode travelling-wave resonantly tuned laser[1]. An inadequacy of these Eqs. (1) is that they are based upon a two-level idealisation of a laser medium whereas it is often argued that a three-level scheme should be used. This means that more realistic systems of laser equations should be of higher dimension. Their reduction to Eqs. (1) is only valid upon a fulfillment of certain conditions. Moreover, the study of three-level models[2,3,4] has displayed a number of common features notintrinsic to the Lorenz model.

It should be noted that these distinctions are not always connected with increasing the dimension. Generally speaking, it is often occurs that the dynamics of multi-dimensional systems, even PDE's, qualitatively resembles dynamics of 2D- or 3D-systems. From the mathematical viewpoint this phenomenon is associated with the existence of a stable invariant low-dimensional manifold on which the dynamics is played. For this, a low-dimensional model with qualitatively similar behaviour can be easily chosen, but to find a low-dimensional model providing a quantitative correspondence is an immeasurably hard problem.

One effective method for solving this problem is the method of normal forms on a center manifold. The normal form correctly describes dynamics in a neighbourhood of an equilibrium state (or a periodic orbit) near an instability threshold. An advantage of the normal form method is that the normal form system is determined by the character of the bifurcation rather than by the specific features of the equations under consideration. It is important to note also that the dimension of the space of solutions of the original equations is not correlated with the dimension of the normal form system, which depends only on the number of characteristic exponents lying on the imaginary axis.

The local bifurcations of codimension less than three generate only one or two-dimensional normal forms, which have been well studied[5,6]. The essential distinction of high-dimensional normal forms is the possibility of chaotic behaviour. It was shown in Ref.[7] that in the case of a bifurcation of an equilibrium with three zero eigenvalues and a complete Jordan block there can arise spiral chaos associated with a homoclinic loop to a saddle-focus. Spiral chaos was also found in the normal form for concrete PDE's describing convection in a rotating layer of salt fluid[8]. We emphasise that the approach based on the reduction to the center manifold is very promising for a rigorous proof of chaos in multi-parameter problems, particularly in magneto- and hydro-dynamics[9,10].

1.3 Quasiperiodicity

The third sort of solution is of great importance in radio-engineering because "there is no radio-location without modulation". Modulated oscillations are mathematically described by quasi-periodic functions. In a dynamical system of finite dimension, the associated quasi-periodic trajectory lies on the surface of an invariant torus whose dimension is given by the number of independent frequencies.

Tori with a quasi-periodic trajectory are well-known in the theory of integrable Hamiltonian systems. However, tori in such systems fill whole volumes in the phase space. The phenomena that a torus in systems of non-conservative origin may be isolated like a Poincaré limit cycle were understood in the early thirties. An invariant torus appears naturally when the original system is periodically forced. Within a framework of the average method that fact was established by Bogulubov and Mitropolskii. An invariant torus may appear via coupling two self-oscillating generators.

The principal physical question to be solved here is often devoted to transition from the synchronization regime to the beating one. A new problem has recently been added to the first, namely, the onset of chaos through torus breakdown. More precisely, how can one explain the following sequence of bifurcations observed regularly in both experiments and numerical simulations: equilibrium state $\rightarrow$ periodic motion $\rightarrow$ quasi-periodic solution with two frequencies $\rightarrow$ resonant periodic orbit $\rightarrow$ chaos. We should nevertheless emphasize that a quasi-periodic solution is not a structurally stable one, i.e. an arbitrarily small C^1-perturbation added to the vector field may lead to a neighbouring system which has no such solution. In the case of a two-dimensional torus, instead of ergodic torus there may appear periodic ones whose periods are longer the smaller the perturbations.

Because we cannot virtually distinguish a quasi-periodic solution from this periodic one, *multi-frequency* oscillations take place as is said in nonlinear language. This is a very important point for natural science when these and only these mathematical results cannot be completely mirrored into nonlinear dynamics theory.

1.4 Strange attractors

We have noted already that a strange attractor is regarded as a mathematical image of chaotic oscillations. The most satisfying deciphering is the following: a strange attractor is supposed to be an attractive limit transitive set, all trajectories of which are unstable.

As we now know, the mathematical literature gives plenty examples of systems with nontrivial behaviour. However, by virtue of some requirements which are very natural for nonlinear dynamics, namely the persistence of properties above under reasonable perturbations, only two groups of genuine strange attractors can be considered: the first are structurally stable and therefore hyperbolic (never discovered in applications so far) and the second are of Lorenz-like type. Their feature is that they are not structurally stable sets. Nevertheless, a Lorenz-like attractor consists of unstable (saddle) trajectories only and this property persists under small perturbations. Despite their wide magnification, Lorenz attractors also have a certain disadvantage because their fractal dimension does not exceed three, regardless of the dimension of the original high-order system.

Most experimental and numerical investigations have shown that the role of the mathematical image of chaotic motions may belong to another curious object, to a *quasi-stochastic attractor* or a *quasi-attractor*. Its uniqueness is that it itself or a quasi-attractor of a neighbouring system contains a structurally unstable Poincaré homoclinic orbit, i.e. an orbit whose stable and unstable invariant manifolds have a tangency. It implies the existence of Newhouse regions in the space of dynamical systems, i.e. regions of structural instability dense everywhere [1,2]. In order to give a complete description of such systems one needs not a finite set of control parameters but an infinite one, compared with just two control parameters in the case of a Lorenz attractor, for example. Moreover, under certain conditions, systems with infinitely many stable periodic orbits are dense everywhere in Newhouse regions [3,4]. This is the second paradox of nonlinear dynamics—in spite of the vividly expressed mathematical character of this phenomena, such subtle effects are totally ignored in experiments and numerical simulations.

2 Models and the Andronov problem

Models of finite-parameter families of differential equations, which determine dynamical flows, are popular objects of research in nonlinear dynamics. These models usually represent generalisations (sometimes a restriction) of dynamical phenomena in natural sciences and technology. Alternatively, a model may result from a purely mathematical investigation. For example, a normal form, which comes from a study of a local bifurcation of an equilibrium state or a periodic orbit (even of a certain homoclinic bifurcation), is an example of the mathematical model.

The problem of the study of models has been formulated by A. Andronov as a two-step process:

1. Partition the parameter space into regions of structural stability and identify a bifurcation set;
2. Divide this bifurcation set into connected components, to each of which corresponds a topologically equivalent structure of trajectories.

According to the method, a *good* model must have a sufficient number of governing parameters in order to carry out a complete analysis of all particular trajectories such as fixed points, limit cycles, homoclinic orbits, etc. which occur in its phase space. However, recent results in the study of systems with complex dynamics, as well of those with homoclinic tangencies, have revealed that the Andronov problem is not always realistic. The problem is that systems with countably many periodic orbits of any degree of degeneracy are dense everywhere in the Newhouse regions [3,4]. We should therefore distance ourselves from the ideology of a "complete description" when dealing with such models and should instead restrict ourselves by studying some specific but more generally-regular properties of the system.

A typical model with chaos is usually such that its parameter space may be subdivided into regions of simple and complex dynamics respectively. In the regions of simple dynamics, where the Andronov setup holds entirely, the primary task is detecting stability regions of equilibrium states and periodic orbits: Let us consider a k-parameter family of a smooth n-dimensional system

$$\dot{x} = X(x, \mu) \tag{1}$$

2.1 Equilibria

We assume that at $\mu = \mu_0$ the system possesses a rough (structurally stable) stable equilibrium state O_{μ_0}. It is obvious that (1) will have stable O_μ for all μ sufficiently close to μ_0. Since the question of the existence of O_μ is reduced to solving the system of equations

$$X(x^*, \mu) = 0 \tag{2}$$

for which the implicit function theorem may be applied, then O_μ depends on μ continuously. By extending in this way the region of existence of O_μ, under the requirement of the stability of O_μ, we can find its stability region. This region is defined by the following inequalities:

$$\Delta_1(\mu) > 0, \ldots, \Delta_{n-i}(\mu) > 0, \Delta_n(\mu) > 0 \tag{3}$$

where Δ_i's are well-known Routh-Hurwitz minors which can be built from the roots of the characteristic equation

$$\lambda^n + b_1(\mu)\lambda^{n-1} + \ldots + b_n(\mu) = 0 \tag{4}$$

2.2 Periodic orbits

We can similarly find the stability region of a periodic orbit. Let our system have a stable periodic orbit $\Gamma_{\mu=0}$ at $\mu = \mu_0$. It will persist for all small $|\mu - \mu_0|$ as well. The stability condition here is the requirement for roots of the characteristic equation of the linearized Poincaré map

$$\Xi(z, \mu) = z^{n-1} + c_1(\mu)z^{n-2} + \ldots + c_{n-1}(\mu) = 0 \tag{5}$$

to lie within a unit circle.

In contrast to the case for equilibrium states, boundaries of the stability region of a periodic orbit may be of two types depending on whether the periodic orbit exists on a boundary or it does not.

In the first case, one can construct the Poincaré map and after that find the stability boundaries which are defined by the following conditions, see [5]:

1. $\Xi(1\,,\,\mu)\;=\;0$ corresponds to one unit multiplier (saddle-node bifurcation);
2. $\Xi(-1\,,\,\mu)\;=\;0$ corresponds to one -1 multiplier (flip bifurcation);
3. $\Xi(e^{\pm i\omega}\,,\,\mu)\;=\;0$ corresponds to a pair of complex conjugate multipliers on a unit circle (Andronov-Hopf bifurcation).

In the second case, where there is no periodic orbit on the boundary, known stability boundaries may be classified as follows:

1. A periodic orbit collapses into a complex focus (~~the length of the periodic orbit goes to infinity while approaching the bifurcation point~~);
2. A periodic orbit turns into a separatrix loop and then a saddle equilibrium point (the period of the periodic orbit tends to infinity) [6];
3. A saddle-node equilibrium arises on a periodic orbit (the same as 2) [6];
4. The "blue sky catastrophe", i.e. when both period and length of a periodic orbit go to infinity while approaching a bifurcation point. This recently-discovered bifurcation is discussed in [7].

The search and the analysis of most of these bifurcations have been well developed and can therefore be implemented in any software; we mention here two packages which are capable of solving these bifurcation problems: LOCBIF (Local Bifurcation Analyzer) and AUTO. The exception is the boundary of the "blue sky catastrophe". Despite the fact that it is a codimension-one boundary, this bifurcation has not yet been found explicitly in applications of nonlinear dynamics.

3 Safe and dangerous boundaries

One has to appreciate the following circumstance: while studying a concrete dynamical system and choosing parameter values, one must take into account not only requirements concerning stability of regimes but also a few others. For instance, it may turn out so that the optimal operating conditions of a device are achieved near its stability boundary.

The second important question is related to the fact that some parameters of the system under consideration may behave in a "quasi-stationary" (evolutionary) manner, thereby leading the system to the stability boundary. The latter means that we have therefore to study the system's behaviour *on the boundary.* The answer is given by the Lyapunov stability theory of critical cases and by the theory of *global* bifurcations. Briefly, the boundaries of stability regions of equilibria and periodic orbits may be of two types:

- "Safe" boundaries are those for which non-deep trespassing over them leads to *small* quantative changes of the system's state.
- "Dangerous" boundaries are those for which arbitrarily small perturbations cause the system's behaviour to change *qualitatively.*

We should notice here that reversing the perturbation brings the system back in the first case but this is impossible in the "dangerous" case. The remainder of this paper elaborates on some particularities of the behaviour of the dynamical system near a stability boundary of codimension one. This allows us to use just one bifurcation parameter. We therefore assume that at $\mu = 0$ the system

$$\dot{x}\;=\;X(x,\mu) \tag{6}$$

lies on a boundary of the stability region, i.e. when $\mu < 0$ it is inside the region and outside if $\mu > 0$.

4 Criteria for safe boundaries [7,8,9]

Case 1

Let O_μ be an equilibrium state and $O_{\mu=0}$ have a single pair of purely imaginary eigenvalues. In this case, the system $X(\mu)$ is written as

$$\begin{aligned}
\dot{x} &= \rho(\mu)x - \omega(\mu)y + [L(\mu)x - a(\mu)y](x^2+y^2) + \ldots + f(x,y,z)z, \\
\dot{y} &= \omega(\mu)x + \rho(\mu)y + [a(\mu)x + L(\mu)y](x^2+y^2) + \ldots + g(x,y,z)z, \\
\dot{z} &= [A(\mu) + h(x,y,z)]z,
\end{aligned} \tag{7}$$

where $\omega(0) \neq 0$, $\rho(0) = 0$ and $\rho(\mu)\,\mu :> 0$ for $\mu \neq 0$; the matrix $A(\mu)$ is a stable one. The corresponding boundary S_1 will be safe if $L(0) < 0$. When μ increases from zero, a unique stable periodic orbit is generated from the complex focus $O_{\mu=0}$ (Fig.1).

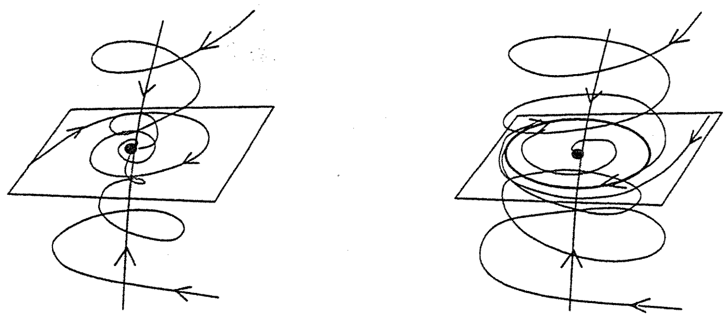

Figure 1: A super-critical Andronov-Hopf bifurcation. An equilibrium state becomes of saddle-focus type.

Case 2

Let Γ_μ be a periodic orbit, one multiplier of which becomes equal to -1 on the stability boundary. The associated Poincaré map T on a cross-section transverse to the periodic orbit may be represented in the following form:

$$\begin{aligned}
\bar{x} &= \rho(\mu)x + a_2(\mu)x^2 + a_3(\mu)x^3 + \ldots + f(x,y,\mu)y, \\
\bar{y} &= [A(\mu) + g(x,y,\mu)]y,
\end{aligned} \tag{8}$$

where $|\rho(0)| = -1$, $|\rho(\mu)| < 1$ for $\mu < 0$ and $|\rho(\mu)| > 0$ for $\mu > 0$, and the eigenvalues of $A(\mu)$ lie within a unit circle. The stability boundary S_2 is safe if the Lyapunov value $L(0) = -2a_3(0) - 2a_2^2(0)$ is negative. As follows from the form of the Poincaré map, the invariant manifold corresponding to $y = 0$ is a Möbius strip with the periodic orbit as its median line. Therefore, for $\mu > 0$ another periodic orbit with a period close to double the period of the original cycle, appears attracting to all nearby trajectories, see Fig.2.

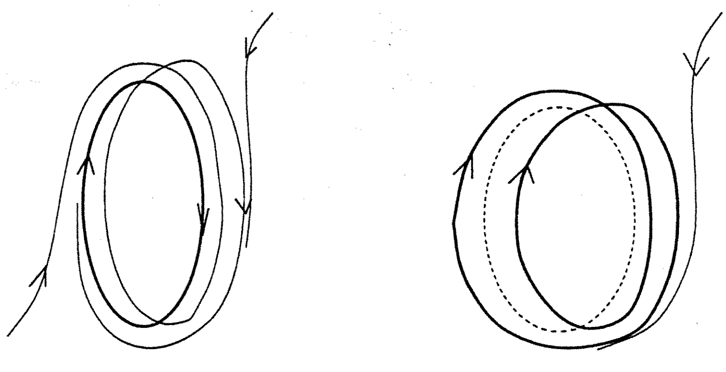

Figure 2: A flip bifurcation.

Case 3

Let $\Gamma(\mu)$ be a periodic orbit which has a pair of multipliers equal to $e^{i\psi(0)}$, where $\psi(0) \neq 0,\ \frac{\pi}{2},\ \frac{2\pi}{3},\ \pi$. The Poincaré map here may be written as

$$\begin{aligned} \bar{x} &= \rho(\mu)[x\cos\psi(\mu) - y\sin\psi(\mu)] + [g(\mu)x - b(\mu)y](x^2+y^2) + \ldots + f(\quad)z, \\ \bar{y} &= \rho(\mu)[x\sin\psi(\mu) + y\cos\psi(\mu)] + [b(\mu)x - g(\mu)y](x^2+y^2) + \ldots + h(\quad)z, \\ \bar{z} &= [A(\mu) + h(x,y,z,\mu)]z, \end{aligned} \tag{9}$$

where $|\rho(\mu)| < 1$ when $\mu < 0$, and $|\rho(\mu)| > 1$ when $\mu > 0$. In this case, the boundary S_3 is safe provided that $g(0) < 0$. Having crossed S_3, a stable 2D-invariant torus appears from the periodic orbit, "the cycle loses its skin" as A. Andronov called this bifurcation, see Fig.3.

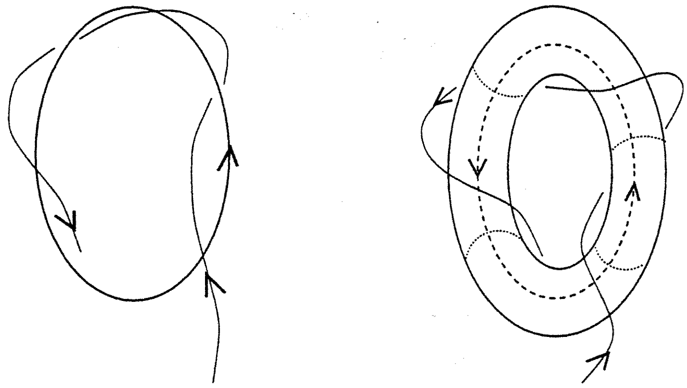

Figure 3: The birth of the invariant torus.

This is the mechanism of a soft transition from self-oscillation to the beating regime.

Case 4

The feature of this case is that the limit of $\Gamma(\mu)$ as $\mu \to 0$ is a contour Γ^* composed of the simplest non-rough equilibrium state of saddle-node type and a its bi-asymptotic separatrix, see Fig.6. The fact that the boundary S_4 (it is defined by one more condition of smoothness of the contour Γ^*) is safe follows directly from the stability of $\Gamma_{\mu=0}$. Beyond the bifurcation point, when $\mu > 0$, the stationary regime persists due to the appearance of the stable point, see Fig.4.

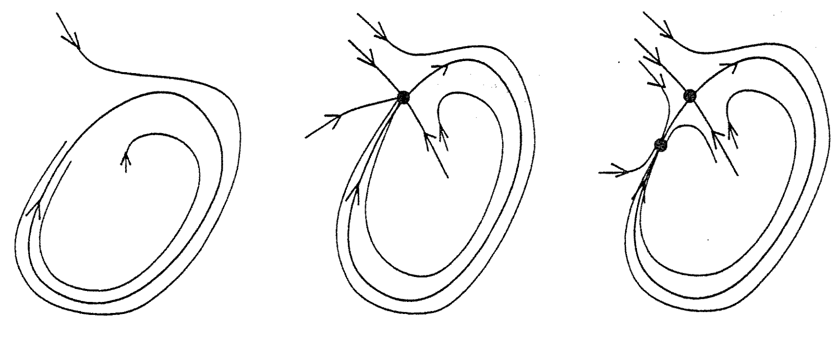

Figure 4: The disappearance of the stable cycle is accomplished with the appearance of the stable node.

Case 5

As we mentioned above, the boundaries of stability regions of periodic orbits contain one more of codimension one which corresponds to the "blue sky catastrophe". We pause here to consider this bifurcation in detail. It may occur in n-dimensional systems where $n \geq 3$.

The feature of the case studied in [9] is that the topological limit of the bifurcating periodic motion L_μ as $\mu \to +0$ contains no equilibrium point (in contrast to first three bifurcations), except for a saddle-node periodic orbit Γ^* disappearing when $\mu < 0$. Γ^* is the simplest saddle-node periodic orbit in the sense that it has only one unit multiplier and the first Lyapunov value is not equal to zero. It means that the stability boundary of L_μ is the codimension-one surface S_{11} corresponding to the saddle-node cycle Γ.

The additional conditions which define S_{11} are of a general type. Let us denote the unstable set of Γ^* as $W^u_{\Gamma^*}$. It is homeomorphic to a semi-cylinder $R^1 \times S^1$. We suppose also that all trajectories in $W^u_{\Gamma^*}$ tend to Γ^* as $t \to +\infty$ and none lies in the strong stable subset $W^{ss}_{\Gamma^*}$, i.e. $W^s_{\Gamma^*} \cap W^{ss}_{\Gamma^*} = \emptyset$. In this case we can claim that $\bar{W}^u_{\Gamma^*}$ is either a torus or a Klein bottle or a non-topological manifold, see Fig.5.

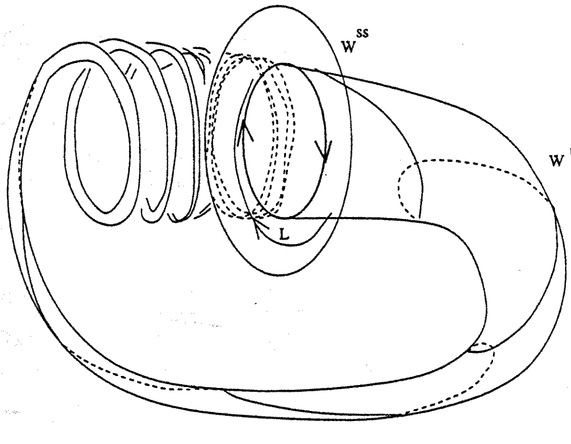

Figure 5: A sketch of the blue sky catastrophe; the shape of the periodic orbit $L(\mu)$ looks like a helix strongly twisted near a saddle-node cycle.

In the last case, the diameter of W^s_Γ, upon coming back to the cycle, shrinks sufficiently for a small $\mu < 0$. At $\mu = 0$, it becomes a stable periodic orbit L_μ whose period and length tend to infinity as $\mu \to +0$. The boundary S_{11} is a safe one, for $\mu > 0$ the orbit Γ is divided into two ones: stable Γ^+_μ and saddle Γ^-_μ. Both are located nearby the ghost of L_μ dissolved. The new regime will be therefore determined by Γ^+_μ. It follows from the fact that the respective Poincaré map is a contracting one (for further details, see [7]).

5 Criteria for dangerous boundaries [8,9]

Case 6

In this case, a topological limit of a periodic orbit $\Gamma(\mu)$ is a separatrix contour comprising a saddle point and a trajectory which is bi-asymptotic to it. A general case of the boundary denoted as S_3 is defined by the following condition [9]: the roots $\rho_1, \ldots, \rho_n$ of the characteristic equation at the saddle are such that $\mathrm{Re}(\rho_i) < 0$ $(i = 1, \ldots, n-1)$, $\rho_n > 0$ and all saddle values $\sigma = \rho_i + \rho_n < 0$ $(i = 1, \ldots, n-1)$.

Thus, one of the trajectories leaving the saddle will belong to Γ^*. The contour Γ^* is unstable because any other trajectory leaving the saddle escapes from a small neighborhood of Γ^*, see Fig.6.

Case 7

Let one eigenvalue of a equilibrium state $O(\mu)$ vanish at $\mu = 0$. The system then can be represented as

$$\begin{aligned} \dot{x} &= R(x, \mu) + f(x, y, \mu)\, y, \\ \dot{y} &= [A(\mu) + g(x, y, \mu)\, y, \end{aligned} \tag{10}$$

where $R(0,0) = R_x(0,0) = 0$. The general case is pointed out by the condition $l_2 = R_{xx}(0,0) \neq 0$. Since we have supposed that the boundary crossing occurs for negative values of μ, then $l_2 > 0$. The associated stability boundary S_7 is a dangerous one: when μ is approaching zero, another equilibrium state is coming

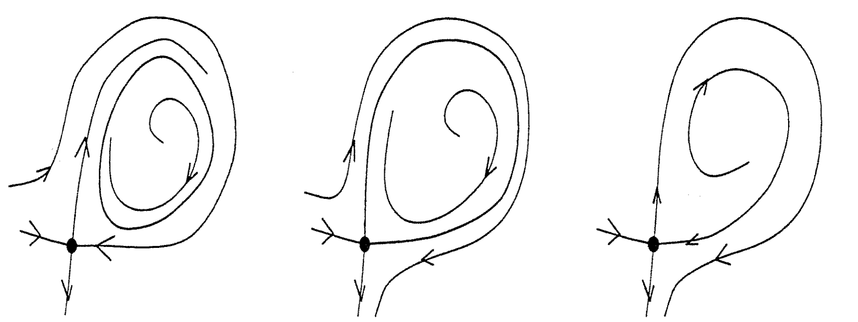

Figure 6: A periodic orbit turns into a homoclinic loop and then a saddle.

to $O(\mu)$ and at $\mu = 0$ both coalesce forming a saddle-node point O^*. When $\mu > 0$ ($R(x,\mu) > 0$) the point disappears and all trajectories leave a neighbourhood of O^*, see Fig.7.

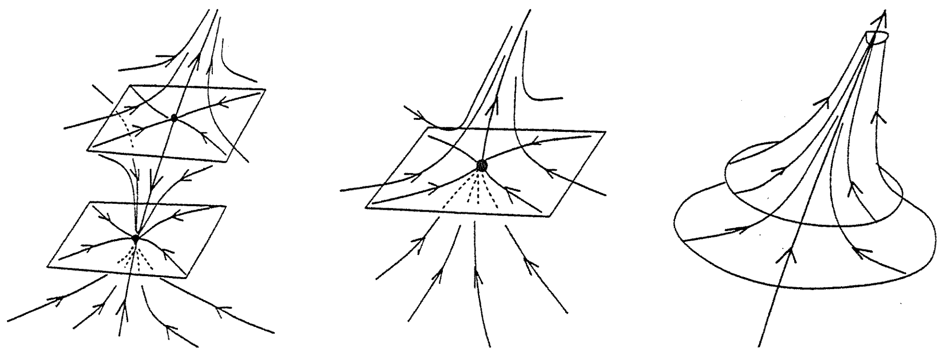

Figure 7: A saddle-node bifurcation in R^3.

Case 8

This is similar to case 1, but with $L(0) > 0$. As $\mu \to 0$, a saddle periodic orbit becomes Γ^*. It makes the boundary S_8 dangerous because the disappearance of the periodic orbits implies that Γ^* has a two-dimensional unstable manifold.

Case 9

This is the same as case 2 but with $L(0) > 0$. The instability is related to a periodic orbit with a double period which turns into Γ^*. When $\mu \geq 0$, an unstable periodic orbit with its unstable manifold homeomorphic to a Möbius strip is born as a result of the bifurcation.

Case 10

Same as case 3 but with $g(0) > 0$. The instability of Γ^* here is due to an unstable torus which shrinks onto it. For $\mu \geq 0$, the cycle has its unstable manifold W^s of dimension three.

Case 11

Let one multiplier of a periodic orbit $\Gamma(\mu)$ become equal to unity at $\mu = 0$. The respective map can be written in the following form

$$\begin{aligned} \bar{x} &= x + R(x,\mu) + f(x,y,\mu)\, y, \\ \bar{y} &= [A(\mu) + g(x,y,\mu)]\, y \end{aligned} \tag{11}$$

The stability boundary is of dimension $(k-1)$ if $l_2 = R_{xx}(0,0) \neq 0$. Under the assumptions above regarding how μ is defined in the system and from $l_2 \neq 0$ it follows that $l_2 > 0$. The dangerous character of boundary S_{11} is caused because $\Gamma(\mu)$ coincides with a saddle periodic orbit as $\mu \rightarrow 0$. The bifurcation orbit Γ^* is of saddle-node type whose unstable manifold W^u is a homeomorphic image of a cylinder $S^1 \times R^+$, where R^+ is a semi-straight-line.

6 Summary

The set of stability boundaries of equilibrium states contains surfaces of three kinds: S_1, S_7 and S_8. The only S_1-like boundaries are safe ones. As for periodic orbits, the set consists of boundaries of nine types: S_6, S_9, S_{10}, S_{11} are dangerous and S_2, S_3, $S_{4,5}$ and $\tilde{S}_1$, $\tilde{S}_2$ are safe (the latter two correspond to gluing a periodic orbit into a complex focus and into a periodic orbit a half-period respectively).

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