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Homoclinic Phenomena in Laser Models

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Abstract—A description of the principal bifurcations which lead to the appearance of the Lorenz attractor is given for the 3D normal form for codimension-3 bifurcations of equilibria and periodic orbits in systems with symmetry. We pay special attention to two bifurcation points corresponding to the formation of a homoclinic butterfly of a saddle with unit saddle index and to a homoclinic butterfly with zero separatrix value.

Keywords—Chaos, Bifurcations, Normal forms, Lorenz attractor, Lasers.

1. INTRODUCTION

The Lorenz model

$$\begin{aligned} \dot{x} &= -\sigma(x-y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy \end{aligned} \tag{1}$$

is well known to be isomorphic with the equations of homogeneously broadened single-mode traveling-wave resonantly tuned laser [1]. An inadequacy of equations (1) is that they are based upon a two-level idealization of a laser medium, whereas it is often argued that a three-level scheme should be used. This means that more realistic systems of laser equations should be of higher dimension. Their reduction to equations (1) is only valid upon fulfillment of certain conditions. Moreover, the study of three-level models [2-5] has displayed a number of common features not intrinsic to the Lorenz model.

It should be noted that these distinctions are not always connected with increasing the dimension. Generally speaking, it often occurs that the dynamics of multidimensional systems, even PDE's, qualitatively resembles dynamics of 2D- or 3D-systems. From the mathematical viewpoint, this phenomenon is associated with the existence of a stable invariant low-dimensional manifold on which the dynamics is played. For this, a low-dimensional model with qualitatively similar behaviour can be easily chosen, but to find a low-dimensional model providing a quantitative correspondence is an immeasurably harder problem.

One effective method for solving this problem is the method of normal forms on a center manifold. The normal form correctly describes dynamics in a neighbourhood of an equilibrium state (or a periodic orbit) near an instability threshold. An advantage of the normal form method is that the normal form system is determined by the character of the bifurcation rather than by the specific features of the equations under consideration. It is important to note also that the dimension of the space of solutions of the original equations is not correlated with the dimension

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of the normal form system, which depends only on the number of characteristic exponents lying on the imaginary axis.

The local bifurcations of codimension less than three generate only one- or two-dimensional normal forms, which have been well studied [6,7]. The essential distinction of high-dimensional normal forms is the possibility of chaotic behaviour. It was shown in [8] that in the case of a bifurcation of an equilibrium with three zero eigenvalues and a complete Jordan block, there can arise spiral chaos associated with a homoclinic loop to a saddle-focus. Spiral chaos was also found in the normal form for concrete PDE's describing convection in a rotating layer of salt fluid [9]. We emphasize that the approach based on the reduction to the center manifold is very promising for a rigorous proof of chaos in multiparameter problems, particularly in magneto- and hydro-dynamics [10,11].

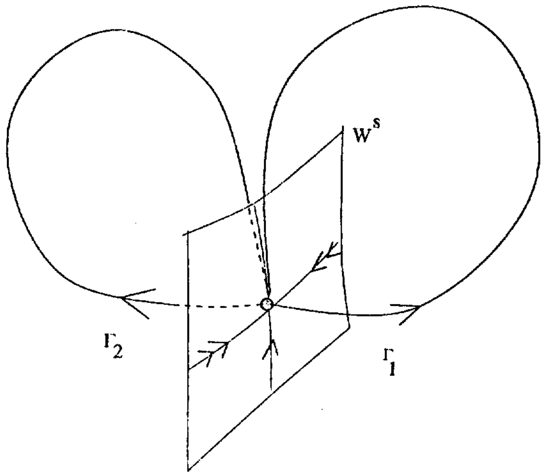


Figure 1. A homoclinic butterfly. One-dimensional separatrices enter the saddle tangentially to each other.

2. THE SUGGESTED SYSTEM

Considered here is the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x(1-z) - Bx^3 - \lambda\, y,\\ \dot{z} &= -\alpha\left(z - x^2\right).\end{aligned} \tag{2}$$

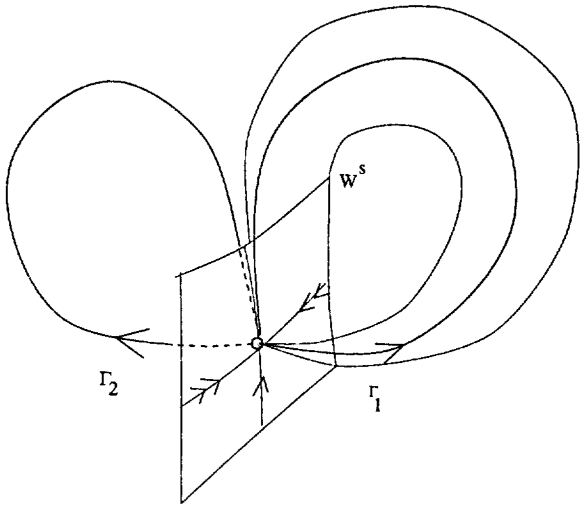
It deserves special interest for several reasons.

(i) This system displays various scenarios of appearance and destruction of chaos and bifurcations of chaotic regimes typical for some three-level laser models.

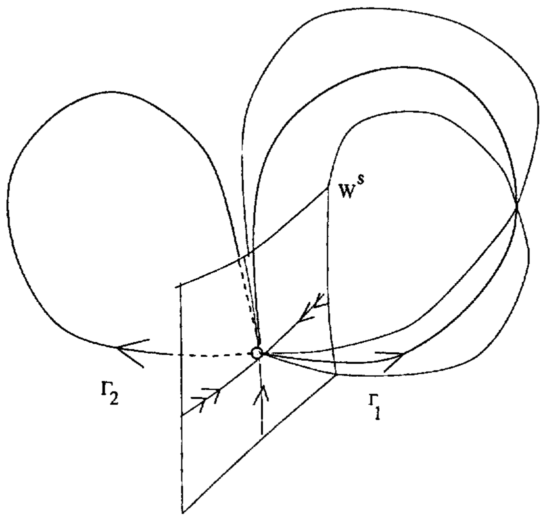
(ii) It is a truncated asymptotic normal form for systems with a triple instability of an equilibrium state or a periodic motion in systems with symmetry [12]. The parameter B is a fixed structural parameter which depends upon coefficients of the original multidimensional system. Values of the parameters α and λ can be arbitrary; they measure proportions between levels of supercriticality. When $B = 0$, equations (2) are the Shimizu-Morioka model [13] which has a strange attractor of the Lorenz type [14] and, as has been established in [15], are the normal form for bifurcations of a zero-intensity state in the LSA model [3,4] (a laser with saturable absorber).

(iii) The Lorenz model (1) can be reduced to system (2) when $r > 1$. The connection between the parameters of the two systems is

$$\alpha = \frac{b}{\sqrt{\sigma(r-1)}}, \qquad \lambda = \frac{(1+\sigma)}{\sqrt{\sigma(r-1)}}, \qquad B = \frac{b}{(2\sigma - b)}. \tag{3}$$



(a)



(b)

Figure 2. A separatrix loop is orientable (a) and nonorientable (b). The invariant manifolds of periodic orbits being born from each loop are homeomorphic to a cylinder in the first case, and to a Möbius strip in the second.

It follows from (3) that the region of the Lorenzian parameters is bounded by the plane $\alpha = 0$ and the surface $\lambda/\alpha = (1/2)((1/B) + 1)$ which tends to $\alpha = 0$ as $B \to 0$.

It is well known that a bifurcation of a homoclinic butterfly is principal in systems of the Lorenz type to which the above-mentioned three-level laser models belong (see Figure 1). This bifurcation does not immediately lead to the appearance of the Lorenz attractor, except for cases enumerated in the theorem of [16]. In systems with symmetry, these cases correspond to homoclinic bifurcations of codimension-2 which are determined by the following conditions.

(i) The saddle index $\gamma = |k_3|/k_1$ is equal to one, while the separatrix value A is positive and less than one. Here $k_2 < k_3 < 0 < k_1$ are the characteristic exponents of the saddle at the origin. In this case, the associated bifurcation curve corresponding to a butterfly is divided by the codimension-2 point into the branches defined by the inequalities $\gamma > 1$ (the stable butterfly) and $\gamma < 1$ (the unstable one).

(ii) The separatrix value A is equal to zero while the saddle index γ is in the interval $(1/2, 1)$. The value A characterizes global properties of the vector field. Depending on the sign of A, a homoclinic loop may be orientable ($A > 0$) as nonorientable ($A < 0$) (Figure 2).
(iii) The homoclinic butterfly is transformed to a figure-8 configuration while the saddle index γ is less then unity [17] (Figure 3).

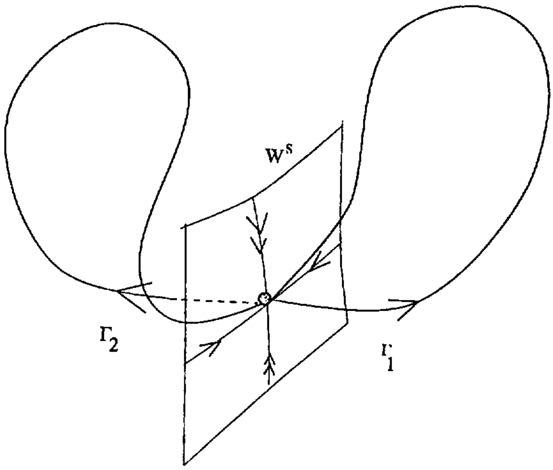


Figure 3. A homoclinic figure-8. One-dimensional separatrices enter the saddle from opposite directions.

We shall show that for $B > -1/3$ in the sector $\alpha > 0$, $\lambda > 0$, there is a region V_{LA} of existence of the Lorenz attractor of system (2). The idea of the proof is to find the bifurcation curve $\mathbf{P}$ in the parameter space (α, λ, B) which corresponds to formation of the homoclinic butterfly with unit saddle index.

The existence of the bifurcation set $H1$ in the parameter space of system (2) was proven in [18] by the method of comparison systems. The proof in this reference can be easily revised in order to confirm the numerically established fact of the existence of the intersection of this set with the surface $\mathbf{M}$ in the parameter space.

It follows from [14] that near the curve $\mathbf{P}$, the size of the region V_{LA} is exponentially narrow. Therefore, in order to show that the existence of the Lorenz attractor is not just in the realm of pure mathematics, we continue numerically the boundaries of the region V_{LA} and show that the region of existence of the Lorenz attractor is sufficiently large and plays as essential role in organizing the global bifurcation portrait.

We give two only one slice (α, λ) fixing $B = 0.0$, since the bifurcation diagrams for other values of B have much in common, see Figure 4.

The points labeled as $\mathbf{P}$ and $\mathbf{R}$ correspond to codimension-2 homoclinic bifurcations: a homoclinic butterfly with $\gamma = 1$ and a double-circuit homoclinic loop with the separatrix value A vanishing, respectively. While moving counter-clockwise around the point $\mathbf{P}$, the bifurcation sequence is the following.

- On $H1^+$, a single-circuit homoclinic butterfly forms; when crossing $H1$, saddle periodic orbits C_1 and C_2 are born together with a nonattractive hyperbolic set Ω.
- On LA1, the separatrix Γ_1 lies on the two-dimensional stable invariant manifold of C_2 (respectively, Γ_2 on the stable manifold of C_1, which is symmetric to C_2), thereby forming an absorbing domain and the set Ω becomes a Lorenz attractor.
- On LC, the Lorenz attractor undergoes an internal crisis: there appears a hole ("lacuna" [19]) containing a saddle symmetric figure-8 periodic orbit C_{12}^*.
- On PF, C_{12}^* becomes stable and a pair of saddle asymmetric periodic orbits C_{12} and C_{21} bifurcates from it.

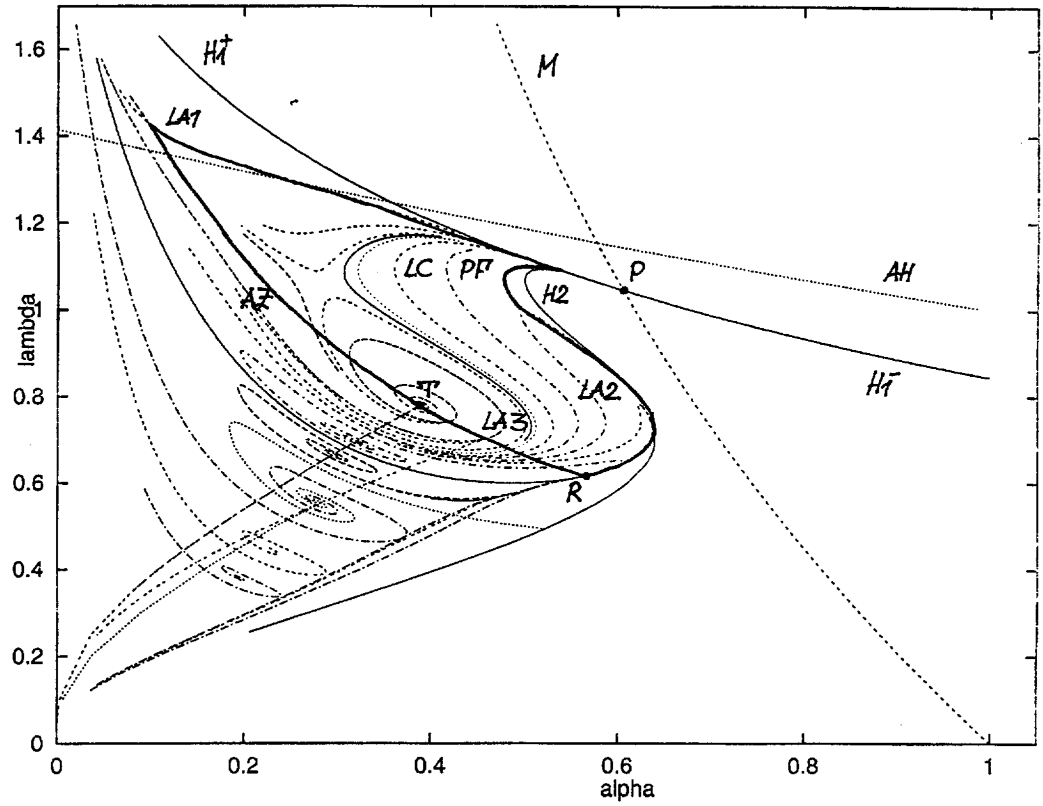
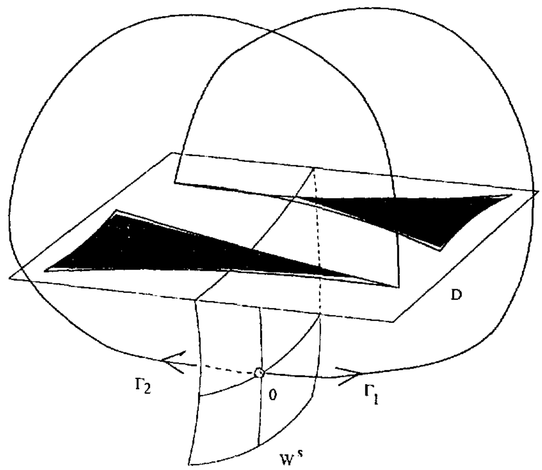


Figure 4. The bifurcation diagrams of system (2) for $B = 0$.

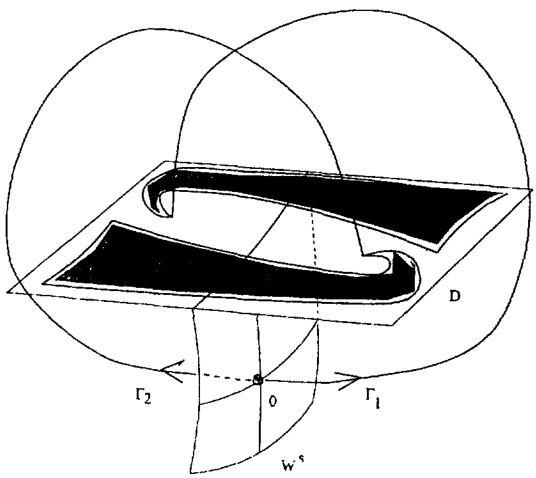
- On LA2, the separatrix Γ_1 lies on the two-dimensional stable invariant manifold of C_{12} and Γ_2.
- On the stable manifold of C_{21}, when crossing LA2, the Lorenz attractor disappears.
- On the curve $H2$, the remaining nonattractive hyperbolic set, driven by C_{12} and C_{21}, flows into the double-circuit homoclinic butterfly.
- On $H1^-$, C_{12}^* becomes the stable single homoclinic butterfly which is formed by two homoclinic loops which are symmetrical to each other; after their splitting, two stable periodic orbits L_1 and L_2 are born.
- On AH, they collapse into nontrivial equilibrium states O_1 and O_2 via the Andronov-Hopf bifurcation.

The curves LA1, LA2, and LA3 are the components of the boundary of the region of existence of the Lorenz attractor. The curves LA1 and LA2 separate regions of simple and Lorenz dynamics. The curve LA3 plays an essentially distinct role and is more completely organized. It includes a countable set $\{R_i\}$ of codimension-2 points corresponding to homoclinic butterflies with zero separatrix value A. As was mentioned above, a region of existence of a Lorenz attractor adjoins to each such point. All of these regions unite into the one bounded by arcs of curves analogous to LA1 and LA2 starting from points R_i (for details, see [20]). Ends of these arcs lie on the curve AZ. Geometrically, this curve corresponds to the formation of distinctive "hooks" in the Poincaré map (see Figure 5). This leads to the appearance of structurally unstable Poincaré homoclinic curves, and consequently, to stable long-period periodic orbits with narrow and tortuous basins.

Attractors containing a complex hyperbolic set together with "weak" stable periodic orbits and homoclinic tangencies are called quasiattractors or "wild" attractors [21]. In contrast to Lorenz-like attractors, the structure of quasiattractors is not clear so far. Moreover, as was shown in [22], a complete description of the structure of quasiattractors cannot be obtained due to uncontrolled bifurcations which are everywhere dense in "wild" regions.



(a)



(b)

Figure 5. The associated Poincaré map for parameter values above (a) and below (b) the curve AZ. Trajectories starting from the return plane next strike it within the shaded areas.

Thus, the curve AZ separates regions of Lorenzian and "wild" dynamics. A transition to quasiattractors from the region of simple dynamics proceeds through a period-doubling cascade. For example, Figure 5b shows the asymmetrical quasiattractor formed via period-doubling of an asymmetric double-circuit stable periodic orbit. When moving deep into the "wild" region, this quasiattractor unites with its mirror image and the symmetrical regime is restored.

The apex of complexity of the regime of wild chaos is achieved when the saddle-foci O_1 and O_2 are involved within it. This occurs between curves F and FF, corresponding, respectively, to homoclinic and heteroclinic contours of the saddle-foci. They are terminated by the codimension-2 T-point which is situated on the boundary LA3 of the region V_{LA} and corresponds to a heteroclinic contour containing all equilibria [22,23]. The existence of such a point may serve as a criterion of both Lorenz-like and wild dynamics in this model. The similar sequences of period doubling of asymmetrical regimes, as well as sudden crises of symmetry, were also found in [2,4].

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