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# On the Boundaries of the Domain of Existence of the Lorenz Attractor*

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Presently in the theory of dynamical systems there is a growing interest in the study of limit sets that are mathematical models of stochastic oscillations. Considerable attention is devoted to the structure of these limit sets, as well as to the study of bifurcations that lead to their appearance. A stimulus for the investigation in this direction has been the discovery of stochastic oscillations in a number of concrete models of practical importance. Among such systems a central place is occupied by the well-known Lorenz model **[8]**, **[10]**:

$$\dot{x} = -\sigma(x - y), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = -bz + xy, \tag{1}$$

which was obtained in the investigation of convective fluid flows and which, it turns out, also describes the dynamics of some laser models **[6]**, disc dynamos **[9]**, and a number of other phenomena.

The basic bifurcations and types of limit sets that may occur in systems similar to the Lorenz model were first studied in **[1]**, **[2]**, where special attention was devoted to the description of the structure of a new type of attracting limit set, which became known as the Lorenz attractor. In this paper we consider bifurcations that define, for system (1), the boundaries of the domain of existence of a strange attractor in the parameter space.

System (1) with $r > 0$ has three equilibrium states, one of which, $O(0, 0, 0)$, is a saddle of type (2, 1). The roots of the characteristic equation at the point $O$ are subject to the inequalities $\lambda_2 < \lambda_1 < 0 < \lambda_3$; the saddle value, equal to $\lambda_1 + \lambda_3$ for $r > (1 + b)(1 + b/\sigma)$, is positive. We denote by $W_0^s$ and by $\Gamma_1, \Gamma_2$ the stable two-dimensional manifold of the saddle $O$ and respectively the two one-dimensional separatrices emanating from $O$.

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Numerical investigations of system (1) showed [1] that for $b = 8/3$, $\sigma = 10$, and $r \simeq 13.92$ both separatrices $\Gamma_1$ and $\Gamma_2$ become doubly asymptotic to $O$ (later this fact was confirmed in [7], also by numerical methods, and subsequently V. N. Belykh [3] gave a rigorous proof of the existence of the separatrix loops $\Gamma_1$ and $\Gamma_2$). When the parameter $r$ passes through the indicated value from each of the separatrix loops there arise saddle periodic motions, $L_1$ and $L_2$, respectively, and, together with them, there also arises a non-attracting limit set $\Omega$, the trajectories of which are into a one-to-one correspondence with the set of doubly infinite sequences of two symbols.

The subsequent evolution of the phase portrait as $r$ is increased is conveniently described using a contactless cross-section, for which one can take the portion of the plane $z = r - 1$ lying between the branches of the hyperbola $xy = bz$. The generalized Poincaré map of the indicated cross-section, generated by the orbits of system (1), possesses the following properties.

On the rectangle $D = \{(x, y) \mid |x| < 1, |y| < 2\}$ there are defined maps $T_1(\mu)\colon D_1 \to D$ and $T_2(\mu)\colon D_2 \to D$, where $D_1 = \{(x, y) \mid |x| < 1,\ 0 < y < 1\}$, $D_2 = \{(x, y) \mid |x| \le 1,\ -1 < y < 0\}$, such that $T_i(\mu)$ has the form

$$\bar{x} = f_i(x, y, \mu), \qquad \bar{y} = g_i(x, y, \mu),$$

where $f_i$, $g_i$ are smooth functions of $x, y, \mu$ that satisfy the conditions

$$f_2(x, y, \mu) = -f_1(-x, -y, \mu),$$
$$g_2(x, y, \mu) = -g_2(-x, -y, \mu)$$

and can be extended by continuity to $y = 0$ so that

$$\lim_{y\to 0} f_i(x, y, \mu) = x_i^0(\mu), \qquad \lim_{y\to 0} g_i(x, y, \mu) = y_i^0(\mu).$$

Moreover,

$$T_1(\mu)D_1 \cap D \subset \Pi_1 = \{(x, y) \mid 1/2 < x < 1, |y| < 1\},$$
$$T_2(\mu)D_2 \cap D \subset \Pi_2 = \{(x, y) \mid -1 < x < 1/2, |y| < 1\}.$$

We note that to the point $P_i(x_i^0, y_i^0)$ there corresponds the first point of intersection of $\Gamma_i$ and $D$, and that the line $S\colon y = 0$ is a connected component of the intersection of $W_0^s$ and $D$ such that all semi-trajectories starting on $D$ have no points of intersection with $D$ for $t > 0$. Taking into account the form of the map generated by the trajectories that pass through a neighborhood of the saddle $O$ [11], we have that in a small neighborhood of the line $S$ the following representation holds:

$$\begin{aligned} f_1(x, y, \mu) &= x_1^0(\mu) + \varphi(x, y, \mu)y^\alpha, \\ g_1(x, y, \mu) &= y_1^0(\mu) + [A(\mu) + \psi_1(x, y, \mu)y^\alpha + \psi_2(x, y, \mu)y^\beta]y^\alpha, \end{aligned} \tag{2}$$

where the functions $\varphi$, $\psi_1$, $\psi_2$ are smooth in the variable $x$, $\psi_1(0, 0, 0) = B \neq 0$, $\psi_2(0, y, \mu) \equiv 0$, $A(\mu)$ is called the separatrix value, and $\alpha$ and $\beta$ are connected with the roots of the characteristic equation as follows: $\alpha = -\lambda_1/\lambda_3$, $\beta = -(-\lambda_2 - \lambda_1)/\lambda_3$. For the Lorenz model in the parameter region characterized by the inequality $r > (2b + 1)(1 + 2b/\sigma)$, one has $\alpha < \beta$, $\alpha < 1/2$.

Let us examine the geometric properties of the maps $T_i$ for $\alpha < 1$ and $A(\mu) > 0$. Let $T \equiv T_i \mid D_i$, $(f, g) = (f_i, g_i)$ on $D_i$, $i = 1, 2$, and let the following inequalities be satisfied:

$$\begin{aligned} &\text{a)}\quad \|f_x\| < 1, \\ &\text{b)}\quad \|g_y^{-1}\| < 1, \\ &\text{c)}\quad 1 - \|f_x\| \cdot \|g_y^{-1}\| > 2\sqrt{\|g_y^{-1}\| \cdot \|g_x\| \cdot \|g_y^{-1} f_y\|}, \\ &\text{d)}\quad \|g_y^{-1} f_y\| \cdot \|g_x\| < (1 - \|f_x\|)(1 - \|g_y^{-1}\|), \end{aligned} \tag{3}$$

where $\|\cdot\| = \sup|\cdot|$ for $(x, y) \in D_1 \cup D_2$.

From representation (2) and inequalities a) and b) in (3) it follows that the image of the rectangle $D$ under the map $T(\mu)$ has the shape of two wedges with zero angle at their apices $P_1$ and $P_2$ (Fig. 1).

Let us consider the basic bifurcations of invariant sets of the map $T(\mu)$ as $\mu$ varies and the accompanying geometrical metamorphoses of the images $TD_1$ and $TD_2$. First of all, the bifurcation consisting in the creation of the separatrix loop $\Gamma_1$ [resp. $\Gamma_2$] corresponds to the position $P_1 \in S$ [resp. $P_2 \in S$]. We shall

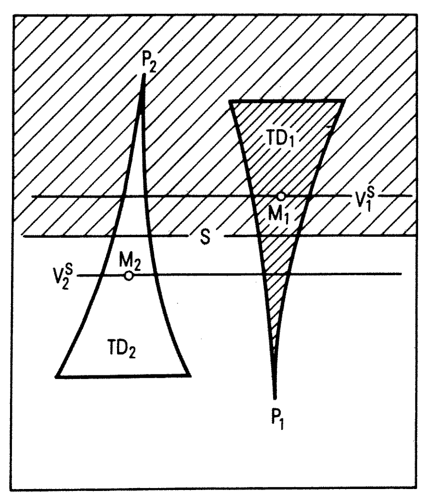


**Figure 1.**

assume that at that moment $\mu = 0$. After this, as the point $P_1$ [resp. $P_2$] crosses into the region $D_2$ [resp. $D_1$], from each separatrix loop there arises a saddle-type periodic motion $L_1$ [resp. $L_2$]. Let $M_i = L_i \cap D$, and suppose that the connected component $V_i^s = W^s(L_i) \cap D$ containing the point $M$ can be written in the form $y = y_i(x, \mu)$. Then the sign of the quantity $R(\mu) = y_2^0(\mu) - y_1(x_2^0, \mu)$ determines the position of the point $P_i$ relative to $V_i^s$. Here $R(\mu) > 0$ for $\mu$ sufficiently small. As the parameter $\mu$ changes, the relative position of the points $P_i$ and curves $y_i(x, \mu)$ also changes, and for some $\mu = \mu_0$ the point $P_i$ lies on the curve $y_i(x_i^0(\mu), \mu)$. At that moment $R(\mu_0) = 0$, and $\Gamma_1$ [resp. $\Gamma_2$] is superposed onto $W^s(L_2)$ [resp. $W^s(L_1)$]. Provided conditions (3) are satisfied, this situation corresponds to the birth of an attracting limit set $\Sigma(\mu)$—the Lorenz attractor. As shown in [**1**], the set $\Sigma(\mu)$ is one-dimensional, the periodic points are dense in it, it consists of two connected components, and it locally has the same structure as the direct product of a Cantor set and a segment. In addition we note that there is a $\mu_1$ such that in the interval $[\mu_0, \mu_1]$ there exists a countable dense set $\{\mu\}$ of bifurcation values of the parameter $\mu$ that correspond to the appearance of separatrix loops $\Gamma_1$ and $\Gamma_2$ of the saddle $O$.

The restrictions (3) imposed on $T(\mu)$ are sufficient conditions for the birth of the limit set $\Sigma(\mu)$ and are of a technical nature, motivated by the method of proof of the existence of a continuous stable foliation of a neighborhood of $\Sigma(\mu)$ into leaves described by Lifschitz functions. Generally speaking, $\Sigma(\mu)$ may remain a strange attractor even when some of the inequalities (3) are violated within certain limits. One of the reasons to which the failure of conditions (3) may be connected is the change of the sign of the separatrix value $A(\mu)$. More precisely, as is readily seen from representation (2), for $\alpha < \beta$, $\alpha < 1/2$ first of all inequality b) fails, which results in the failure of the complete expansiveness condition and leads to the possibility of the appearance of stable periodic or fixed points of the map $T(\mu)$. Geometrically the change of the sign of $A(\mu)$ is accompanied by the formation of bends of the images $TD_1$ and $TD_2$ near the points $P_1$ and $P_2$, respectively (Fig. 2); these in turn lead, when $\mu$ changes further, to the appearance of Smale horseshoes and structurally unstable homoclinic points and hence, according to [**5**], to the possibility of the appearance of stable periodic motions. The birth of stable periodic motions leads in the general case to the destruction of the Lorenz attractor.

For the Lorenz model, to the moment when the separatrix $\Gamma_1$ [resp. $\Gamma_2$] superposed onto the stable manifold $W_2^s$ [resp. $W_1^s$] of the periodic motion $L_1$ [resp. $L_2$] there corresponds a bifurcation curve $l_a$ in the plane of the parameters $(r, \sigma)$ for $b = 8/3$ (Fig. 3). Let $l_a^+$ denote the connected component of this curve specified by the property that for the points on $l_a^+$ the Poincaré map of the plane $z = r - 1$ into itself satisfies the conditions (3). Then $l_a^+$ belongs to the boundary

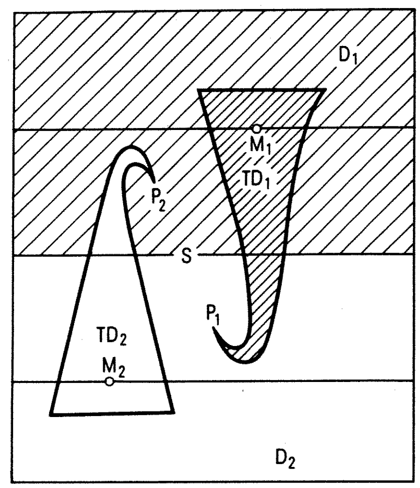


**Figure 2.**

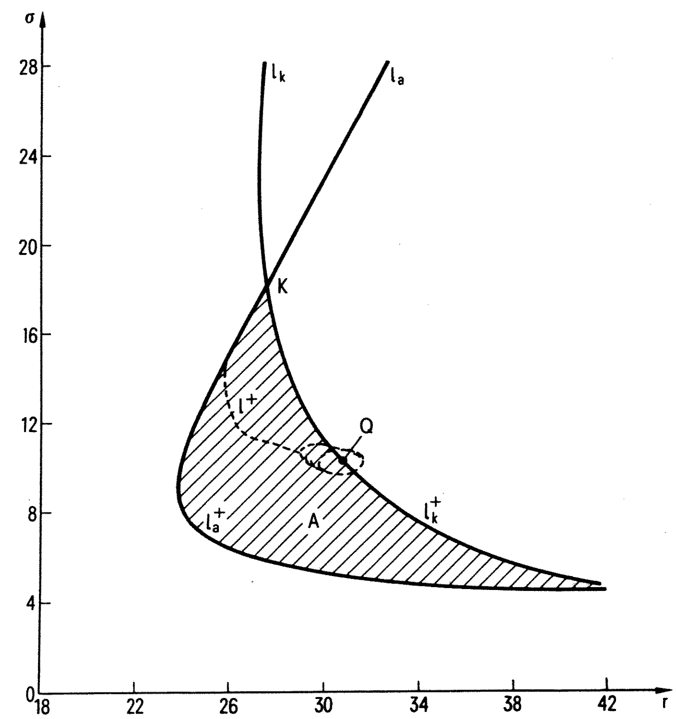


**Figure 3.**

of the domain of existence of the strange attractor, and the birth of the attractor takes place as one crosses $l_a^+$ from the left to the right in the plane $(r, \sigma)$.

Let $U$ be some domain that is intersected by the component $L$ and at whose points the inequalities (3) hold. Then, as we remarked earlier, for parameter values in $U$ there exists a countable, everywhere dense bifurcation set $\{l_\alpha\}$ of curves $l_\alpha$ that correspond to the formation of separatrix loops $\Gamma_1$ and $\Gamma_2$, as well as a continuum $\{\gamma_\alpha\}$, locally homeomorphic to the direct product of the Cantor discontinuum and a segment, of bifurcation curves $\gamma_\alpha$, to whose points there corresponds the situation in which the separatrices $\Gamma_1$ and $\Gamma_2$ are superposed onto the stable manifold $W^s(\Omega)$ of some hyperbolic set $\Omega$ in the Lorenz attractor.

Any of the connected components of the bifurcation sets $\{l_\alpha\}$ and $\{\gamma_\alpha\}$ can be extended beyond the domain $U$, either to the point at infinity or to points of higher codimension that belong to the boundary of these components. Such points can be points that correspond to the formation of a separatrix contour $L$, consisting of the saddle-foci $O_1$ and $O_2$, the coinciding one-dimensional separatrices of the saddle $O$ and a saddle-focus and the trajectory $\Gamma_0$, which belongs to the transverse intersection of two-dimensional manifolds of the saddle and the saddle-focus. Moreover, on each of the components $l_\alpha$ and $\gamma_\alpha$ there can be points, corresponding to the vanishing of the separatrix value, which, generally speaking, are branching points, i.e., they belong to the closure of other bifurcation curves.

In the process of the numerical construction of a part of the curves in the set $\{l_\alpha\}$ we isolated a curve $l^*$ in the shape of spiral winding toward a point $Q$ (located approximately at (30.4, 10.2)). That point belongs to a bifurcation set of codimension two and corresponds to the creation of a separatrix contour $L$. According to [**4**], one has the following:

**Theorem 1.** *On the curve $l^*$ there exists a countable set of points $C_i$, at each of which the separatrix value vanishes and from which emanate bifurcation curves $l_i^+$ and $l_i^-$, corresponding to a structurally unstable periodic motion with a multiplier equal to* $+1$ *and* $-1$, *respectively.*

We also mention the following theorem.

**Theorem 2.** *In any sufficiently small neighborhood of the point $Q$ there exists a countable set of points $Q_i$, accumulating at the point $Q$, to each of which there corresponds a separatrix contour with the same properties as the original one.*

From the study of the map generated by the trajectories in a neighborhood of the separatrix contour $L$ it follows that in the parameter space there exists a countable $\{M_\alpha\}$ of curve arcs $M_\alpha$ to whose points there corresponds the vanishing of the separatrix value $A(\mu)$. As we already mentioned, this corresponds to the beginning of the creation of a "bend" or the formation of a Smale horseshoe.

The extension of each of the segments $M_\alpha$ shows that $M_\alpha$ belongs to one of the curves $l_k$. Furthermore, since the points $C_i \in l_k$ and, for a suitable direction of crossing through $l_k$, periodic motions are born at the point $C_i$, it follows that a part of the curve $l_k$ belongs to the boundary of the domain of existence of the Lorenz attractor.

The curves $l_a$ and $l_k$ have the shape of hyperbolas intersecting at the point $K(27.7, 17.7)$ (see Fig. 3). That point divides each of the curves $l_a$ and $l_k$ into two connected components $l_a^+$, $l_a^-$ and $l_k^+$, $l_k^-$, respectively. The components $l_a^+$ and $l_a^-$ differ in that on crossing $l_a^+$ [resp. $l_a^-$] in the space of parameters $(r, \sigma)$ so that $r$ increases each connected component of $W^u(\Omega) \cap D$ intersects $W^s(L_1) \cap D$ or $W^s(L_2) \cap D$ along at most one trajectory [resp. at most two trajectories].

The components $l_a^+$ and $l_k^-$ of the respective bifurcation curves $l_a$ and $l_k$ point out in the space of parameters $(r, \sigma)$ the domain $A$ of existence of the Lorenz attractor. When one exits $A$ through the boundary $l_a^+$ the Lorenz attractor disappears, but leaves after it a nontrivial attracting hyperbolic set. The boundary $l_k^-$ separates the domain of existence of the Lorenz attractor from a domain at the points of which the Lorenz system has a quasi-attractor. In that domain the complex structure of the partition of the phase space into trajectories is preserved (by the foregoing analysis, there exist homoclinic trajectories); however, in addition, there may exist structurally unstable homoclinic trajectories and stable periodic motions of long period that have a sufficiently small basin of attraction.

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