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BIFURCATION AND PREDICTABILITY ANALYSIS OF A LOW-ORDER ATMOSPHERIC CIRCULATION MODEL

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A comprehensive bifurcation analysis of a low-order atmospheric circulation model is carried out. It is shown that the model admits a codimension-2 saddle-node-Hopf bifurcation. The principal mechanisms leading to the appearance of complex dynamics around this bifurcation are described and various routes to chaotic behavior are identified, such as the transition through the period doubling cascade, the breakdown of an invariant torus and homoclinic bifurcations of a saddle-focus. Non-trivial limit sets in the form of a chaotic attractor or a chaotic repeller are found in some parameter ranges. Their presence implies an enhanced unpredictability of the system for parameter values corresponding to the winter season.

1. Introduction

Atmospheric physics provides some of the most striking and convincing examples of complex dynamical behavior observed in natural sciences, from oscillations and multistability to deterministic chaos (Nicolis & Nicolis [1987]). Most of the real world models of the atmosphere involve a large number of variables and parameters. A great deal of effort has therefore been devoted to the development of truncated versions, with the hope that the study of such low-order dynamical systems would allow one to disentangle the principal mechanisms leading to the well-known unpredictability of weather and climate (Saltzman [1978]). Most of the work carried out in this area has been limited to numerical simulations in which a particular set of parameter values or a particular pathway in parameter space is chosen at the outset, thereby leaving aside a great part of the richness contained in the original equations. Our purpose in the present paper is to undertake a detailed qualitative analysis, supplemented with numerical simulations, of a low-order general atmospheric circulation model proposed by Lorenz, capable of unfolding the entire variety of characteristic behaviors displayed by the system.

The model equations are (Lorenz [1984, 1987]):

$$\begin{aligned}\frac{dX}{dt} &= -Y^2 - Z^2 - aX + aF\,,\\ \frac{dY}{dt} &= XY - bXZ - Y + G\,,\\ \frac{dZ}{dt} &= bXY + XZ - Z\,.\end{aligned} \tag{1}$$

where variable X represents the strength of the globally averaged westerly current (identified through the geostrophic relation with the cross-latitude temperature difference) and Y, Z are the strength of the cosine and sine phases of a chain of superposed waves. The unit of the variable t is equal to the damping time of the waves, estimated to be five days. The terms in F and G represent thermal forcings: F stands for the cross-latitude heating contrast, whereas G accounts for the heating contrast between oceans and continents. Finally, parameter b stands for the strength of the advection of the waves by the westerly current. Hereafter we shall treat F, G as control parameters and set $a = \frac{1}{4}$ and $b = 4$.

In Sec. 2 we carry out the linear stability analysis of the fixed point solutions of Eqs. (1) and identify a codimension-2 bifurcation point. The principal local bifurcations around this point are analyzed in Sec. 3. Homoclinic bifurcations are considered and the existence of strange sets (attracting and repelling) established in Sec. 4. Bifurcations of periodic orbits are considered in Sec. 5 and the main conclusions are drawn in Sec. 6.

2. Fixed Points and Linear Stability

An efficient method for revealing the behavior of a dynamical system is to determine the fixed points (steady state solutions); carry out linear stability analysis around them thus determining the points in parameter space where local bifurcations can be expected; explore the possibilities of global bifurcations by identifying, through control of several parameters, high codimension bifurcation points; and finally complete the picture obtained by this analytic investigation by computer simulations.

We first deal with the fixed points (x_0, y_0, z_0) of Eqs. (1),

$$\begin{aligned}0 &= -y_0^2 - z_0^2 - ax_0 + aF\,,\\ 0 &= x_0y_0 - bx_0z_0 - y_0 + G\,,\\ 0 &= bx_0y_0 + x_0z_0 - z_0\,.\end{aligned} \tag{2}$$

From the second and third equation we find

$$\begin{aligned}y_0 &= \frac{G(1-x_0)}{1-2x_0+(1+b^2)x_0^2}\,,\\ z_0 &= \frac{bGx_0}{1-2x_0+(1+b^2)x_0^2}\,,\end{aligned} \tag{3}$$

Substituting into the first Eq. (2) one finds

$$(1+b^2)x_0^3 - [2+(1+b^2)F]x_0^2 + (1+2F)x_0 + \left(\frac{G^2}{a} - F\right) = 0 \tag{4a}$$

Introducing the new parameters

$$B = \frac{1}{1+b^2}\,, \quad G' = \frac{G^2}{a} - \frac{F}{1+b^2}\,.$$

and the linear transformation

$$x_0 = \overline{x} + \frac{2B+F}{3}\,;$$

we may write Eq. (4a) in the canonical form of a cubic

$$\overline{x}^3 + p\overline{x} + q = 0\,, \tag{4b}$$

where

$$p = B(1+2F) - \frac{(2B+F)^2}{3}$$

$$q = \frac{B(1+2F)(2B+F)}{3} + G' - \frac{2(2B+F)^3}{27}$$

The discriminant of Eq. (4b) is

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}\,.$$

The curves labeled SN in Fig. 1, determined by $\Delta = 0$, separate the regions in the parameter plane (F, G) where system (1) possesses either one or three fixed points. The precise site of the cusp of the curve SN is determined by the simultaneous vanishing of p and q. This occurs when

$$G = \frac{2\sqrt{12}b\sqrt{ab}}{3(1+b^2)}\,, \quad F = \frac{1+\sqrt{3}b}{1+b^2}$$

The stability of the fixed points is determined by the eigenvalues ω of the Jacobian matrix of Eqs. (1), which satisfy

$$\begin{vmatrix} -a-\omega & -2y_0 & -2z_0 \\ y_0 - bz_0 & x_0 - 1 - \omega & -bx_0 \\ by_0 + z_0 & bx_0 & x_0 - 1 - \omega \end{vmatrix} = 0$$

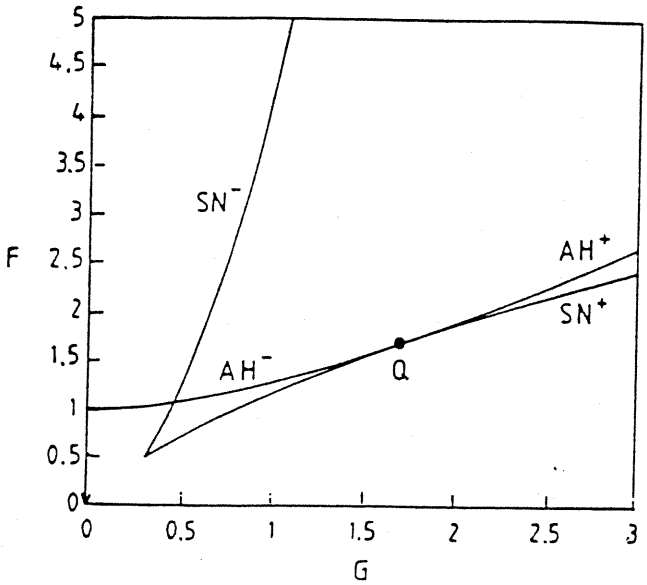


Fig. 1. The (F, G)-parameter plane of system (1) near the codimension-2 point.

leading to the characteristic equation

$$\omega^3 + u\omega^2 + v\omega + w = 0\,, \tag{5}$$

where

$$\begin{aligned} u &= a - 2(x_0 - 1)\,, \\ v &= (x_0 - 1)^2 - 2(x_0 - 1)a \\ &\quad + b^2 x_0^2 + 2(y_0^2 + z_0^2)\,, \\ w &= a(x_0 - 1)^2 + ab^2 x_0^2 \\ &\quad + 2(y_0^2 + z_0^2)(1 - x_0 - b^2 x_0) \end{aligned} \tag{6}$$

Equation (5) has one zero root along the curves SN and one pair of purely imaginary roots along curves AH^+ and AH^- of Fig. 1. The first set is the locus of limit point bifurcations with the exception of the cusp in which a pitchfork bifurcation takes place; the second set is the locus of Hopf bifurcations. When $u = 0$ and $w = 0$, Eq. (5) has simultaneously one zero root and one pair of purely imaginary roots. By substituting the values of x_0, y_0, z_0 into Eqs. (3)–(4a) one identifies a point ($G^* \approx 1.6829$, $F^* \approx 1.6840$) in parameter space, denoted by Q in Fig. 1, which is a codimension-2 bifurcation point (Guckenheimer & Holmes [1983]). It is well known from bifurcation theory that a system in the vicinity of such a point may give rise to very rich dynamical behavior which may subsequently extend far away in parameter space. In particular:

— *Quasi-periodic solutions* can be generated through a secondary bifurcation mechanism.

— Global bifurcations such as *homoclinic bifurcations* can take place (Shil'nikov [1969]), giving rise to closed infinite period orbits that are biasymptotic to a fixed point of the saddle-focus type, i.e. they converge to it in the double limit $t \to \pm\infty$.

These phenomena are analyzed in detail in the next two sections.

3. Local Bifurcations Around the Codimension-2 Point

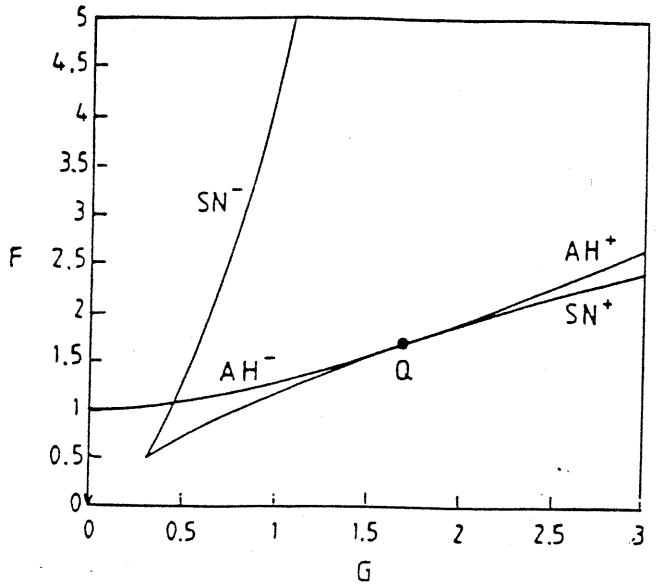
It is well known that the analysis of the principal bifurcations near the codimension-2 point identified above is based on the consideration of a truncated normal form which, in cylindrical coordinates, may be written as (Gavrilov [1978]; Guckenheimer & Holmes [1983]):

$$\begin{aligned} \dot{r} &= r(\mu_1 + \alpha z + z^2) \\ \dot{z} &= \mu_2 + z^2 \pm r^2\,, \\ \dot{\psi} &= 1\,. \end{aligned} \tag{7}$$

where the radial and angle variables r and ψ are associated with the occurrence of a pair of complex conjugate eigenvalues in the characteristic equation.

System (7) is invariant with respect to rotation around the z-axis, and its trajectories lie on integral surfaces determined by trajectories of the first two equations, which are decoupled from the third one. This reduces the problem to the study of bifurcations occurring in a planar system. We note that fixed points of this planar system with $r = 0$ correspond to fixed points of the 3D system (1), fixed points with $r \neq 0$ correspond to periodic motions, and structurally stable limit cycles correspond to invariant tori.

Depending on the sign of a and that in front of r^2 the topological structure of the trajectories of system (7) may belong to one of four different types in a neighborhood of the fixed point. Accurate numerical analysis reveals that our case corresponds to $\alpha < 0$ and "+" in the second equation; this happens to be the richest case from the point of view of dynamical systems theory. The corresponding bifurcation diagram is given schematically in the upper part of Fig. 2. The lower part of the figure lists the change of phase portraits in the (z, r) plane in a small neighborhood U of the bifurcation point Q, as one moves counter-clockwise around this point.



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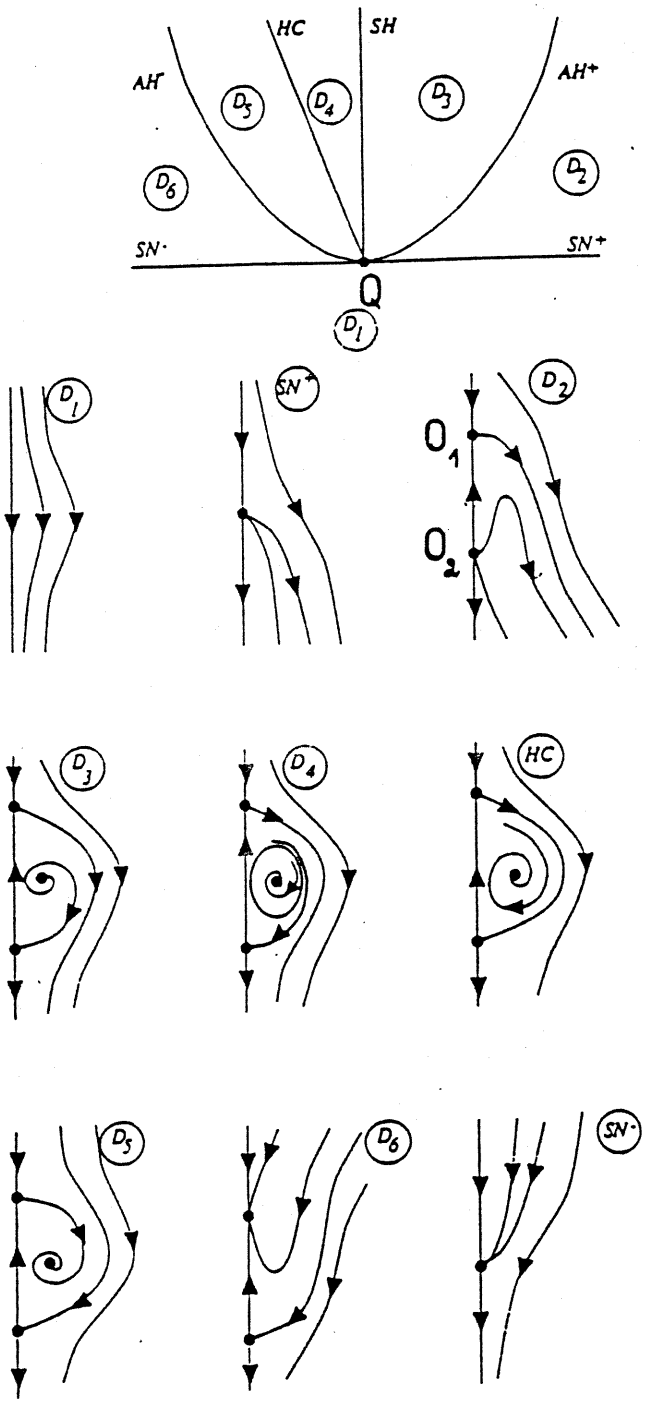


Fig. 2. The unfolding of the planar system (7).

For parameter values in the region D_1, all trajectories leave neighborhood U as $t \to +\infty$ and are attracted to a large-amplitude fixed point as shown in Fig. 3.

On SN^+ there is a structurally unstable fixed point of saddle-node type with one zero and a pair of complex-conjugate eigenvalues with positive real part.

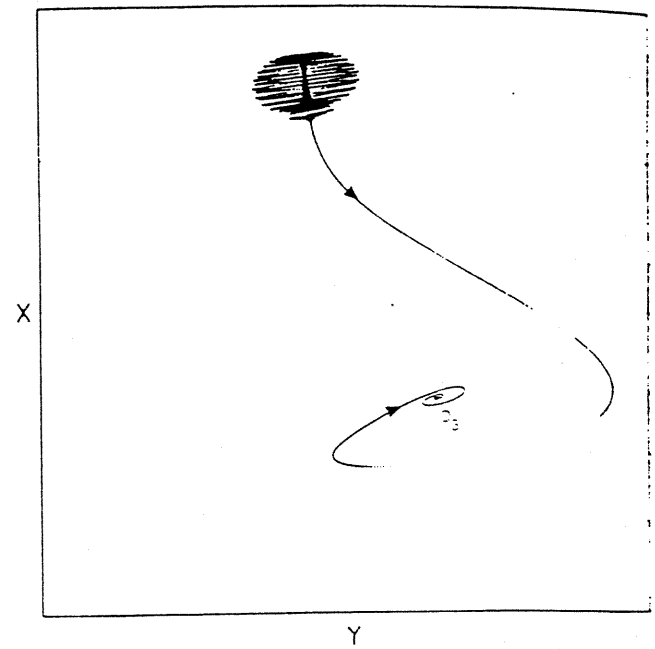


Fig. 3. The stable focus is the unique limit set for the parameter values below the saddle-node curve SN, $F = 1.7$, $G = 1.65$.

When coming into the region D_2 this fixed point is divided into two: a saddle-focus O_1 of $(1,2)$-type, i.e. with one-dimensional stable and the two-dimensional unstable invariant manifold, and a totally unstable (repelling) focus O_2.

On the curve AH^+, the characteristic equation at O_2 possesses a pair of purely imaginary and one positive characteristic exponent. Since the first Lyapunov value is positive at O_2, on crossing the curve AH^+ the fixed point becomes a saddle-focus of $(2,1)$-type and a repelling periodic orbit L is generated from it in region D_3 through a subcritical Andronov–Hopf bifurcation. Figure 4 shows a trajectory started near this unstable periodic orbit L.

On entering the region D_4, the periodic orbit L becomes stable and a totally unstable invariant torus is born from it, see Fig. 5. On the curve HC, the torus becomes a heteroclinic connection between both saddle-foci O_1 and O_2, see Fig. 6.

While approaching the curve AH^- in region D_5 the stable periodic orbit L shrinks into the saddle-focus O_2, which becomes stable in region D_6 through a supercritical Andronov–Hopf bifurcation.

On the branch SN^-, the stable focus O_2 coalesces with the saddle-focus O_1 thereby forming a structurally unstable fixed point of the saddle-node type, which disappears on entering the region D_1.

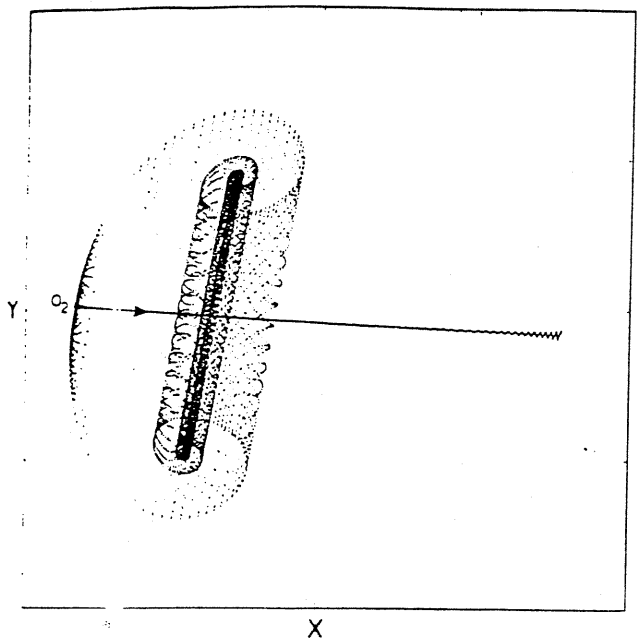


Fig. 4. The totally unstable periodic orbit is the edge of the 2D stable invariant manifold of the saddle-focus O_2 in the reverse time for $F = 1.77$, $G = 1.8$.

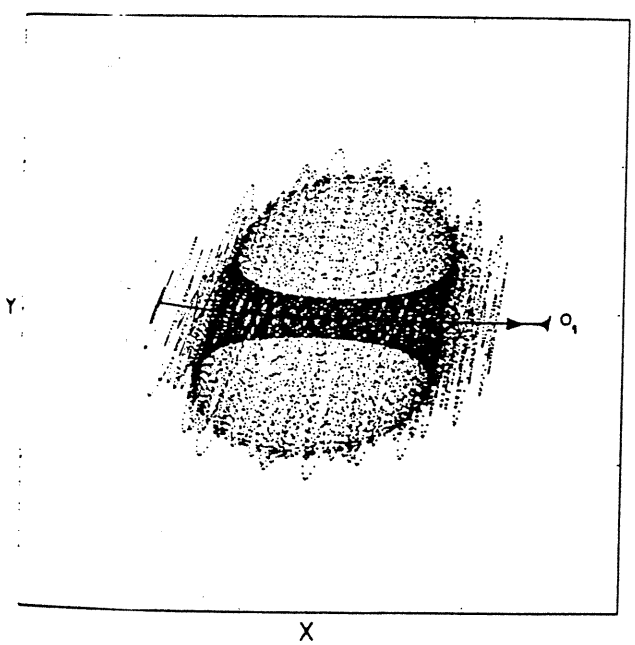


Fig. 5. The totally unstable torus for $F = 1.8$, $G = 1.65$.

The next problem to be solved is to continue the bifurcation loci in the parameter plane (F, G) away from the small neighborhood U of the codimension-2 point, since we intend to study how this point organizes a global bifurcation portrait. Using the

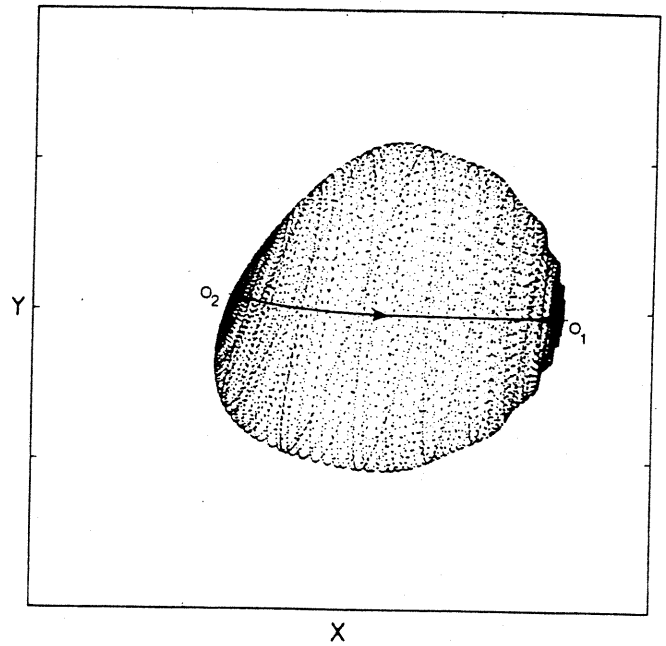


Fig. 6. The repelling torus becomes a heteroclinic connection between saddle-foci O_1 and O_2 for $F = 1.8$, $G = 1.5$.

software LCBIF (Khibnik *et al.* [1993]) we have numerically extended the bifurcation curves AH and SH which correspond to the primary and the secondary Andronov–Hopf bifurcations, respectively, see Fig. 1. In doing so, a subtle problem arises in connection with the behavior of homo- and heteroclinic solutions. This problem is addressed in the following section.

4. Homoclinic Bifurcations

The bifurcations described in the previous section are subjected to the condition of invariance with respect to rotations around the z-axis. The straight line $r = 0$ is then an integral curve and, in the case where both O_1 and O_2 are saddles, their common one-dimensional separatrix. Moreover, in such symmetrical systems, both 2D stable and unstable invariant manifolds of the saddles O_1 and O_2 may either coalesce or have no common points. As far as general two-parameter systems are concerned 1D separatrices of saddle points may coincide at particular parameter values only, whereas 2D separatrices of saddles may cross each other along some trajectories for an open set of parameter values. One further case is also possible, whereby a 1D-separatrix of a saddle-focus (with characteristic exponents $\lambda_1 > 0$,

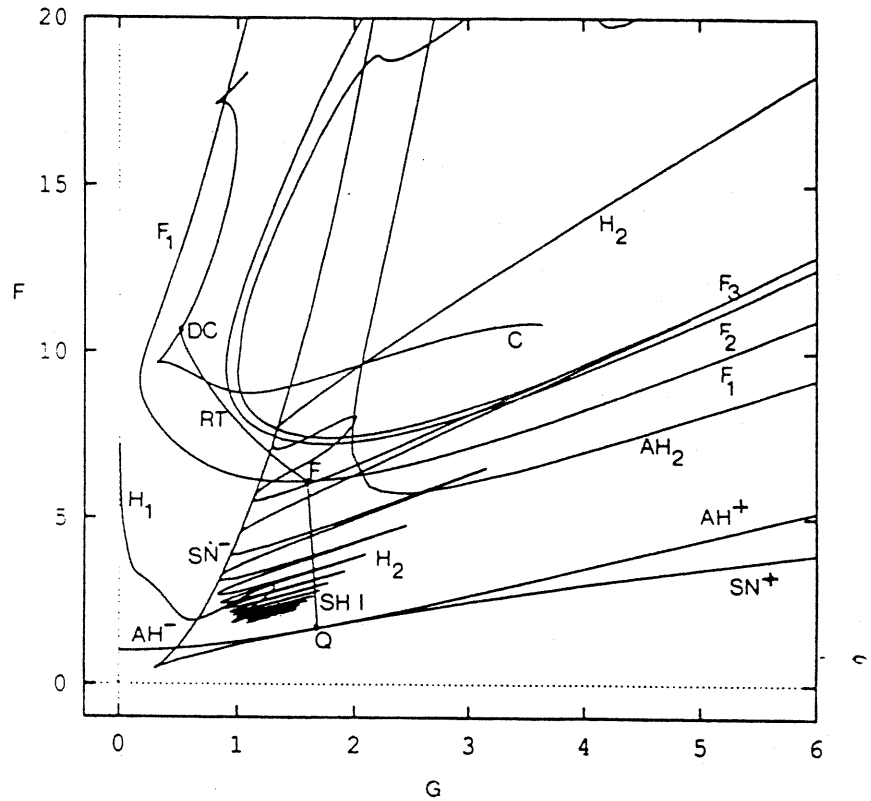


Fig. 7. The bifurcation diagram of system (1) continued beyond the immediate vicinity of the codimension-2 point.

$\lambda_2 = \overline{\lambda}_3$, $\mathrm{Re}\,\lambda_2 < 0$) lies in its 2D stable manifold thereby forming a homoclinic loop. If the so-called saddle value $\sigma = \lambda_1 + \lambda_2$ is positive, there are then infinitely many saddle periodic orbits in a fixed neighborhood of such a loop, as follows from the theorem by Shil'nikov [1969].

There are at least two curves of homoclinic bifurcations emanating from a codimension-2 point of the kind identified in Sec. 2 (Gaspard [1987]), which are associated with homoclinic bifurcations of saddle-foci O_1 and O_2. We have numerically calculated these curves labeled by H_1 and H_2 in the parameter plane (F, G), which correspond to homoclinic loops to saddle-foci O_1 and O_2, respectively, see Fig. 7. Since the one-dimensional unstable (stable) separatrix of saddle-focus O_1 (O_2) passes very closely by the saddle-focus O_2 (O_1) for parameter values near the codimension-2 point, for numerical reasons these bifurcation curves could only be detected far away from it. Nevertheless, the bifurcation parameter values found at particular points confirm that the oscillations occurring in the homoclinic locus decay rapidly when the codimension-2 point is approached (Gaspard [1993]; Kirk [1990]).

Along the bifurcation curve H_2 the saddle value σ at O_2 is negative. This means that only a stable periodic orbit may appear from the homoclinic loop to the saddle-focus O_2. This is valid for the points where the curve H_2 coincides with the curve SN^-. At each such point, the system (1) possesses a saddle-node fixed point with a homoclinic curve to it (Shil'nikov [1963]; Lykyanov [1982]). Since the real part of both non-zero characteristic exponents of the saddle-node is negative, a stable periodic orbit of long period is created in homoclinic bifurcation, see Fig. 8.

A scenario for the appearance of a complex invariant set from a saddle-focus was suggested in Shil'nikov [1991]; Ovsyannikov & Shil'nikov [1992]. The first step is the transition from a stable focus to a saddle-focus via a supercritical Andronov–Hopf bifurcation. In our case it occurs when crossing the curve AH^-, below which the stable focus is the only limit set. Above AH^- the periodic orbit L (see Sec. 3 and Fig. 4) is the edge of the two-dimensional unstable invariant manifold W^u of the saddle-focus. The Floquet multipliers of the periodic orbit become complex but remain within the

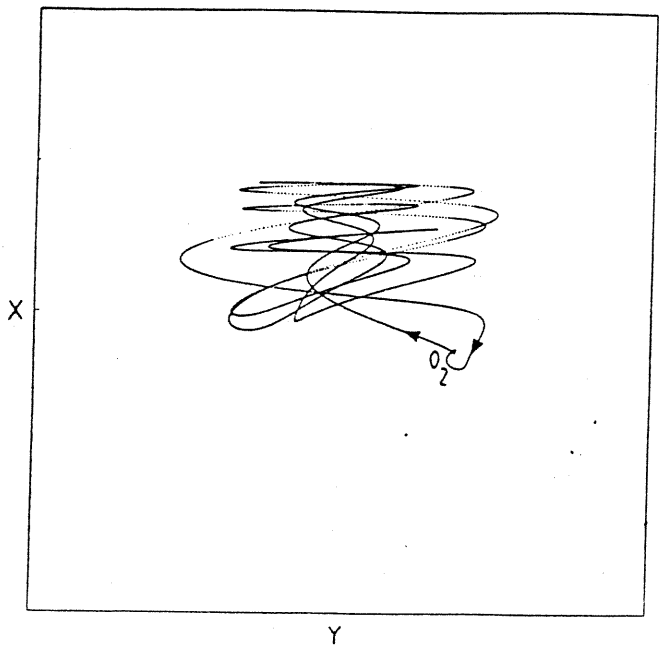


Fig. 8. A long period stable periodic orbit appearing from a saddle-node for $F = 5.0$, $G = 1.077$.

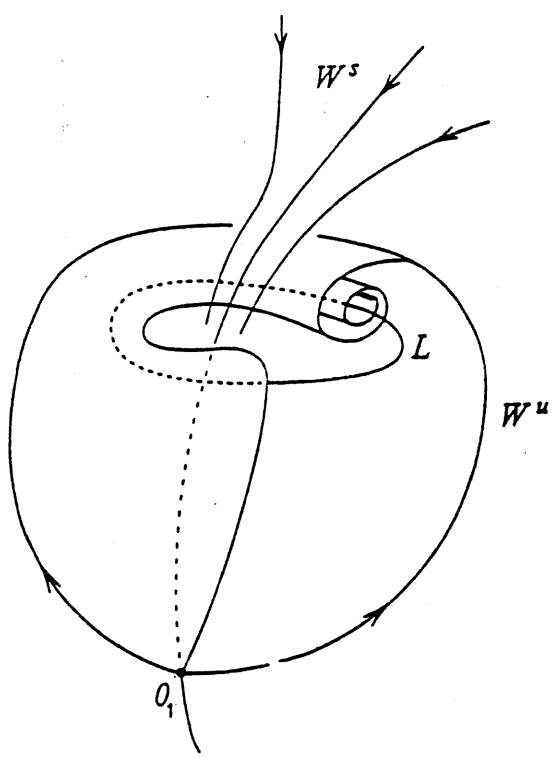


Fig. 9. The formation of a whirl-pool repeller.

unit circle; this causes the unstable manifold to begin to spiral onto the periodic orbit thereby forming a "whirlpool", Fig. 9. The size of scrolls increases and on the curve H_1 of Fig. 7 the 2D unstable in-

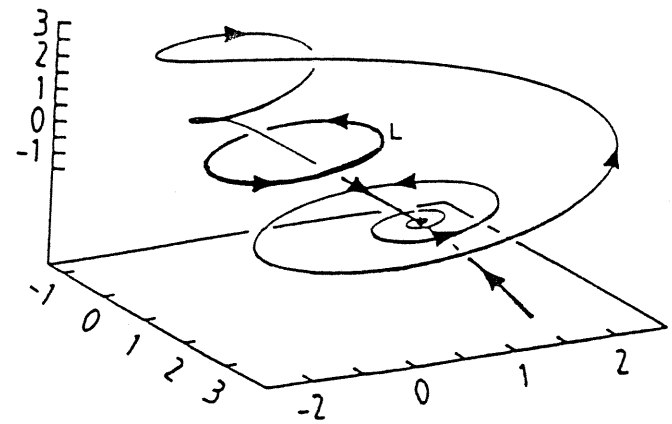


Fig. 10. A homoclinic loop to the saddle-focus O_1 on the curve H_1, $F = 4.0$, $G = 0.08$.

variant manifold W^u touches the 1D stable invariant manifold W^s, Fig. 10.

Such a non-trivial limit set may inherit the stability of the stable periodic orbit L thereby forming a spiral (or screw-like) attractor. Actually in our case, the complex limit set arose before the cycle has lost its stability and is attractive backward in time, i.e. it is a repeller. One can show that the main cause for this phenomenon is the alternating sign of the divergence of the vector field defined by Eqs. (1). Figures 11(a)–11(c) give three views of the repeller for different pairs of values of the parameters F and G.

5. Bifurcation of Periodic Orbits and Invariant Tori

A strange attractor (repeller) of the spiral type in a three-dimensional system is as a rule not a conventional set but a *quasi-chaotic* attractor (repeller) due to the presence of structurally unstable Poincaré homoclinic orbits (Ovsyannikov & Shil'nikov [1992]; Afraimovich & Shil'nikov [1983]). The existence of the latter may lead to saddle-node periodic orbits and therefore to stable periodic orbits which usually have very weak and tortuous basins of attraction (Newhouse [1979]). The situation in the case of a system with a sign-alternating divergence may be even more sophisticated, namely, saddle-node bifurcations of periodic orbits may cause the appearance of not only stable periodic orbits but also of totally unstable (repelling) ones. As proved in Gonchenko, Turaev & Shil'nikov [1992]; Gonchenko, Turaev & Shil'nikov [1993], a complete description of a quasi-attractor is unattainable due to infinitely many

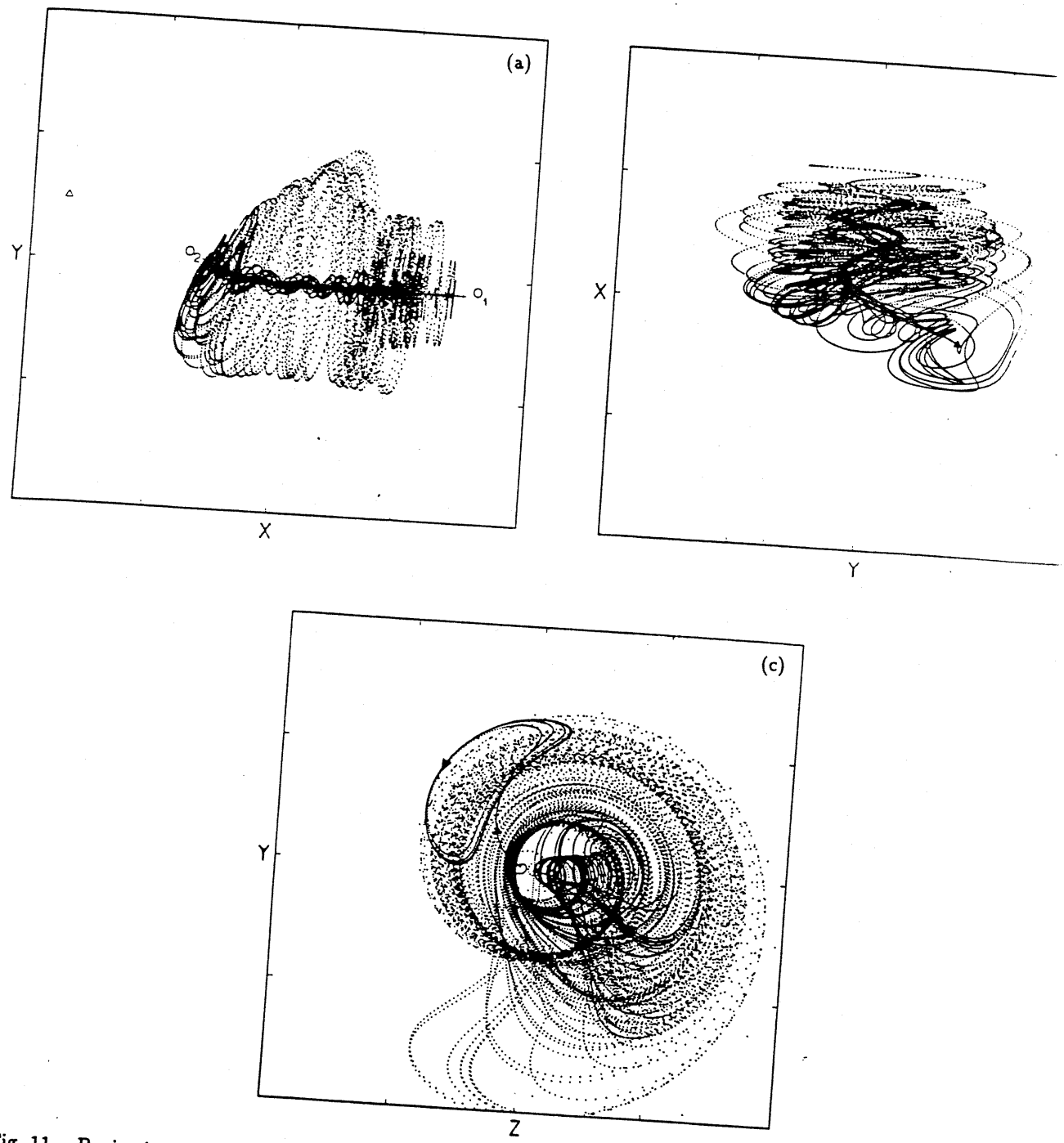


Fig. 11. Projections of a chaotic repeller for (a) $F = 2.5$, $G = 1.4$; (b) $F = 5$, $G = 1.0778$; and (c) $F = 10$, $G = 3$.

non-controlled bifurcations of various types: homoclinic, saddle-node, period doubling, etc. We therefore limit ourselves to point out certain types of behavior for specific parameter values.

Although stable periodic orbits of sufficiently large periods are practically invisible within a spiral attractor or a repeller, some bifurcations of periodic orbits of a short period may be detected numerically. The periodic orbit L loses its stability when crossing the curve F_1 in the parameter space (F, G) (see Fig. 7) via the flip bifurcation (one multiplier is equal to -1). This curve is the first in a period doubling cascade (the next two bifurcation curves F_2, F_3 are also shown in Fig. 7) which leads to the appearance of a Feigenbaum-like set in the phase space of system (1). Such a limit set may be both an attractor and a repeller depending on parameters. The boundary between these regions passes

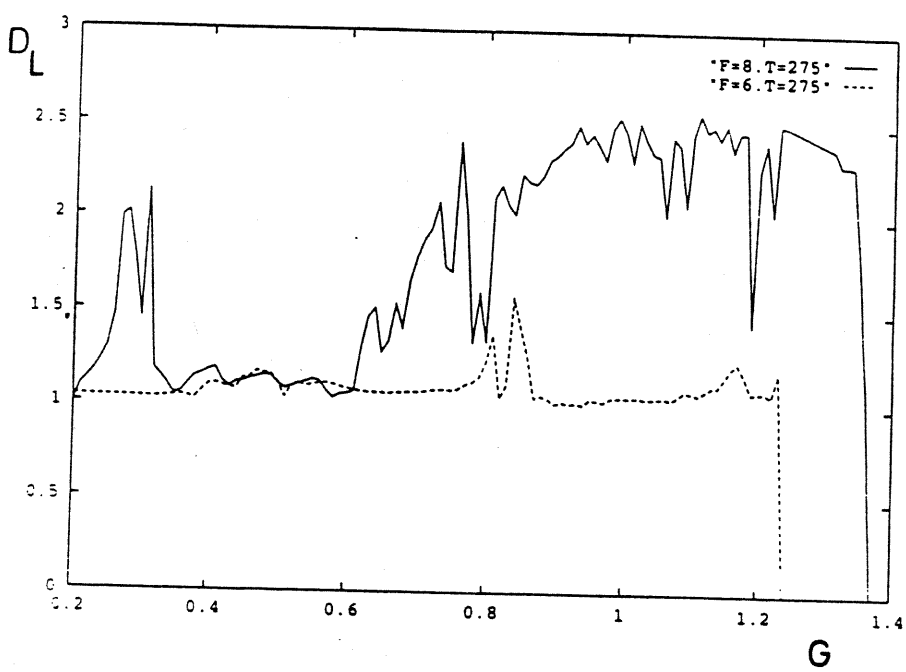
through the point F on the curve F_1, where the structurally unstable periodic orbit has two multipliers equal to -1. This point, where a resonance in the ratio 1:2 takes place, terminates the curve SH of secondary Hopf bifurcations. It is also the terminal point of the curve RT on which a torus emerges from the doubled periodic orbit that appeared in the flip bifurcation on F_1. Along the curve RT, the phase ψ of the multipliers $e^{\pm i\psi}$ of this structurally unstable periodic orbit is varied from π to 0. It implies that any Arnol'd resonance tongue can adjoin to RT from above. In this region, quasiperiodicity is the typical behavior of bifurcations of system (1). The other end point DC of curve RT corresponds to a 1:1 resonance, when the periodic orbit has two multipliers equal to $+1$. This point also lies on the curve C where the periodic orbit undergoes a saddle-node bifurcation.

6. Discussion

We have seen that Eqs. (1) can generate a variety of behaviors of considerable complexity, culminating in the appearance of strange sets of the quasi-attractor or quasi-repeller type. Given enough time, a trajectory on such a set will pass close to the basins of attractions of its stable periodic orbits. This will result in transiently a stage of regular periodic-like behavior and will complicate considerably the applicability of the currently available algorithms of computation of dimensionalities and Lyapunov exponents (Mayer Kress [1986]; Abraham *et al.* [1989]).

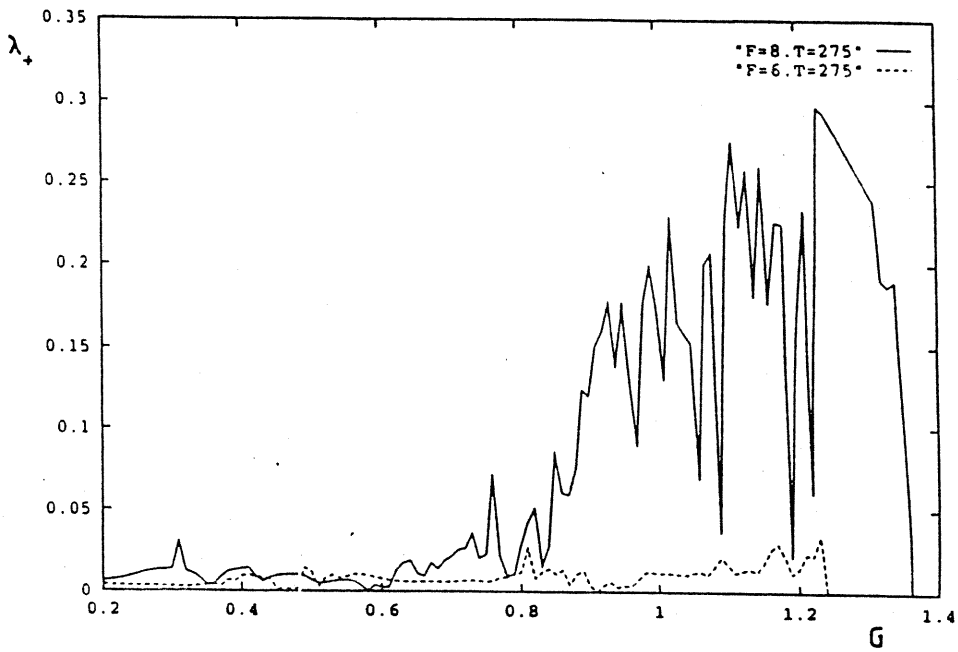
A useful tool for the qualitative analysis of systems of the above type is the Lyapunov dimension D_L of a limit set as defined by the Kaplan–Yorke formula (Kaplan & Yorke [1979]). Figure 12(a) represents the result of the numerical computation of D_L for $F = 8$ and $F = 6$ and for G in the range $(0.2, 1.4)$. We see a clearcut signature of chaos in certain regions, i.e. for $F = 8$ and G around 1. On the other hand, for $F = 6$ the dimension drops frequently to values between 1 and 2. This is likely to reflect the influence, during the integration time, of the above mentioned stages of periodic-like behavior interrupting intermittently the chaotic evolution.

The results summarized in Fig. 12 are further confirmed by the computation of the system's largest Lyapunov exponent λ_+, depicted in Fig. 12(b) as a function of the parameter G. We also see from this figure that the dynamics for $F = 8$ is more unpredictable than for $F = 6$. Now, according to the interpretation of F and G (Lorenz [1984, 1987]) a winter season corresponds, on the average, to larger



(a)

Fig. 12. (a) Lyapunov dimension of the chaotic repeller for $F = 6$ and $F = 8$ and for G in the interval $(0.2, 1.4)$, obtained from the integration of Eqs. (1) during $T = 275$ time units. (b) The largest Lyapunov exponent of the chaotic repeller under the same conditions as in Fig. 12(a).



(b)

Fig. 12. (*Continued*)

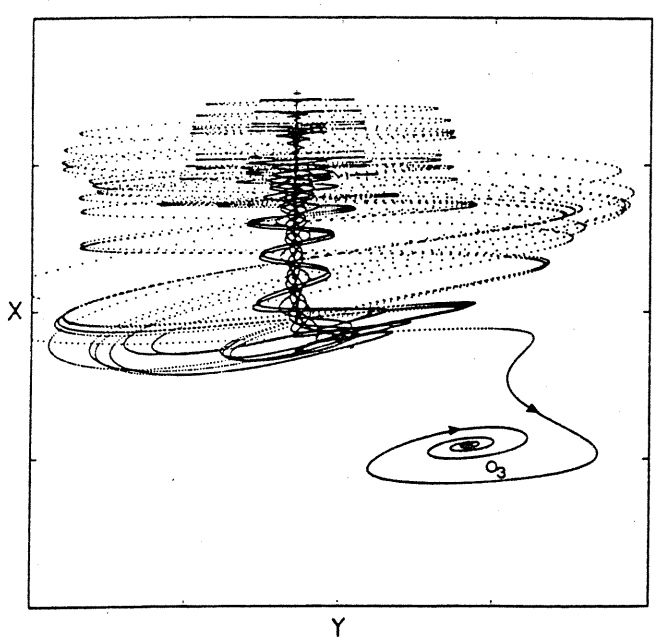


Fig. 13. The basin of the stable node is extended as the parameter G is increased, $F = 2.5$, $G = 1.4$.

values of these parameters than the summer one. Our results suggest, therefore, that winter circulation tends to be more unpredictable than summer circulation. This is in agreement with the recent analysis of a multi-level primitive equation model reported in the meteorological literature (Buizza & Palmer [1994]).

For values of F, G beyond the range of Figs. 12(a) and 12(b) the stable fixed point arising from the saddle-node bifurcations on curve SN^- (Figs. 1 and 7) captures almost all trajectories if the system is integrated for a very long time, see Fig. 13. Nevertheless, the behavior of a trajectory starting near a saddle-type solution [see e.g. Fig. 11(a)] may look chaotic for a substantial period of time, as it will tend to follow the unstable manifold of this solution during a time scale of the order of the inverse of its largest Lyapunov exponent.

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bifurcations of equilibria and periodic orbits, the graphics and integration package DSTOOL (Back *et al.* [1992]) and the continuation package AUTO (Doedel [1986]).

References

Abraham, N., Albano, A., Passamante, A. & Rapp, P. eds [1989] *Measures of Complexity and Chaos* (Plenum, New York).

Afraimovich, V. S. & Shil'nikov, L. P. [1983] "Strange attractors and quasi-attractors," in *Nonlinear Dynamics and Turbulence*, Barenblatt, G. I., Iooss, G. & Joseph, D. D. eds. (Pitman, NY), pp. 1–28.

Back, A., Guckenheimer, J., Myers, M., Wicklin, F. & Worfolk, P. [1992]. "DSTOOL: Computer-assisted exploration of dynamical systems," *Notices of the AMS* **39**, 303–309.

Bao, Z. & Nicolis, C. [1988] Report, University of Brussels (unpublished).

Buizza, R. & Palmer, T. [1995] "The singular vector structure of the atmospheric general circulation," *J. Atmos. Sci.*, **52**, 1434–1456.

Doedel, E. [1986] AUTO: Software for continuation and bifurcation problems in ordinary differential equation, C.I.T. Press, Pasadena.

Gaspard, P. [1987] "Tangences homocliniques dans les systèmes dynamiques dissipatifs," Ph.D. dissertation, Université Libre de Bruxelles.

Gaspard, P. [1993] "Local birth of homoclinic chaos," *Physica* **D62**, 94–121.

Gavrilov, N. [1978] "On some bifurcations of an equilibrium state with one zero and a pure imaginary pair of characteristic exponents," in *Methods of Qualitative Theory of Differential Equations*, Leontovich-Andronova, E. A., ed., pp. 33–40 (in Russian).

Gonchenko, S. V., Turaev, D. V. & Shil'nikov, L. P. [1992] "On models with a structurally unstable homoclinic Poincaré curve," *Sov. Math. Dokl.* **44**, 422–426.

Gonchenko, S. V., Turaev, D. V. & Shil'nikov, L. P. [1993] "On models with non-rough Poincaré homoclinic curves," *Physica* **D62**, 1–14.

Guckenheimer, J. & Holmes, P. [1983] *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer, New York).

Kaplan J. & Yorke, J. [1979] "Chaotic behavior of multidimensional difference equations," *Lecture Notes in Mathematics* (Springer, NY), pp. 204–227.

Khibnik, A., Kuznetsov, Y., Levitin, V. & Nicolaev, E. [1993] "Continuation techniques and interactive software for bifurcation analysis of ODEs and iterated maps," *Physica* **D62**, 360–371.

Kirk, V. [1990] "Destruction of tori in dissipative flows," Ph.D. dissertation, University of Cambridge.

Lorenz, E. [1984] "Irregularity: A fundamental property of the atmosphere," *Tellus* **A36**, 98–110.

Lorenz, E. [1987] "Deterministic and stochastic aspects of atmospheric dynamics," in *Irreversible Phenomena and Dynamical Systems Analysis in Geosciences*, Nicolis, C. & Nicolis, G., eds. (Reidel, Dordrecht), pp. 159–179.

Lorenz, E. [1990] "Can chaos and intransitivity lead to interannual variability?," *Tellus* **A42**, 378–389.

Lykyanov, V. [1982] "On bifurcations of dynamical systems with a separatrix loop to a saddle-node," *Differential Equations* **58**, 1493–1506 (in Russian).

Mayer Kress, G., ed. [1986] *Dimensions and Entropies in Chaotic Systems* (Springer, Berlin).

Newhouse, S. E. [1979] "The abundance of wild hyperbolic sets and non-smooth sets for diffeomorphisms," *Publ. Math. IHES* **50**, 101–151.

Nicolis, C. & Nicolis, G., eds. [1987] *Irreversible Phenomena and Dynamical Systems Analysis in Geosciences* (Reidel, Dordrecht).

Ovsyannikov, I. & Shil'nikov, L. P. [1992] "Systems with a homoclinic curve to the multi-dimensional saddle-focus and spiral chaos," *Math. Sbornik* **2**, 415–443.

Saltzman, B. [1978] "A survey of statistical-dynamical models of the terrestral climate," *Advances in Geophysics* **20**, 184–304.

Shil'nikov, L. P. [1963] "On some cases of birth of periodic orbits from singular trajectories," *Math. Sbornik* **61**, 433–466 (in Russian).

Shil'nikov, L. P. [1969] "On a new type of bifurcation of multi-dimensional dynamical systems," *Soviet Math. Dokl.* **10**, 1368–1371.

Shil'nikov, L. P. [1991] "The theory of bifurcations and turbulence," Part 1, *Selecta Mathematica Sovietica* **2**, 45–53.

Shil'nikov, A., Turaev, D. & Shil'nikov, L. P. [1993] "Normal forms and Lorenz attractors," *Bifurcation and Chaos* **3**, 1123–1139.