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ON THE NONSYMMETRICAL LORENZ MODEL*

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The structure of the bifurcation set for non-symmetric Lorenz model is studied. Regions of existence of simple and complex dynamics are pointed out. It is shown that there exist two qualitatively distinct scenarios of transition to chaos.

Consider the model

$$\dot{x} = -\sigma(x-y)\,, \quad \dot{y} = -y + rx - xz + R\,, \quad \dot{z} = -bz + yx \tag{1}$$

which originates from the study of the liquid dynamics [Yorke & Yorke, 1981; Varandas & Mendonca, 1987] within a convective loop at the Lorenz parameters $b = \frac{8}{3}$ and $\sigma = 10$. Since this model possesses the symmetry $(x, y, R) \leftrightarrow (-x, -y, -R)$, it is sufficient to study only for $R \geq 0$. In Fig. 1 a bifurcation diagram in the domain $0 \leq r \leq 32$, $0 \leq R \leq 13$ is presented. The points A, B, C, D, E are well known and are located at the values $r_A = 1.0$, $r_B = 13.92$, $r_C = 24.06$, $r_D = 24.77$, and $r_E = 30.98$, respectively.

In the domain D_1 there exists one stable equilibrium state O_1. To the right of the curve $S_1^1 \cup F \cup S^2$ there are three equilibrium states O_1, O_2 and O. Crossing the curve S_1^1 the equilibrium states O_2 and O are born from a saddle-node. Hence, O_2 is a stable equilibrium state and O is a saddle of the type (2, 1). Crossing the curve S_1^2 the equilibrium states O_2 and O are born from an equilibrium state of the saddle-saddle type and so O_2 is a saddle of the type (1, 2), and O a saddle of the type (2, 1). The point $F(6.077, 7.190)$ is a point of codimension two corresponding to a nonrough equilibrium state with two zero roots of the characteristic equation. This is a limit point for the two bifurcation curves S_2 and S_4. If by Γ_1 and Γ_2 we denote the trajectories starting from O so that as t tends to ∞ the trajectory Γ_2 tends to the equilibrium point O, then the curve S_2 will correspond to the homoclinic loop $\overline{\Gamma}_2 = \Gamma_2 \cup O$ (Fig. 2a). On the curve S_4 the equilibrium state O_2 loses its stability. At this very instant the saddle periodic motion L_2 emerging from $\overline{\Gamma}_2$ tightens into the equilibrium state O. The bifurcation curve S_3 corresponds to the homoclinic loop $\overline{\Gamma}_1 = \Gamma_1 \cup O$ (Fig. 2b). Upon crossing the curve S_3 to the right the saddle periodic motion L_1 emerges from it. The curve S_8 is a stability boundary of equilibrium state O_2. On this boundary the motion L_1 tightens into the equilibrium state O_1. On the curve S_5, there is the inclusion $\Gamma_1 \in W^s_{L_1}$ (Fig. 2c). This curve separates the domain of simple dynamics (to the left) from the complex one (to the right). The curve S_6 corresponds to the inclusion $\Gamma_1 \in W^s_{L_2}$ (Fig. 2d) and the curve S_7 to the inclusion $\Gamma_2 \in W^s_{L_1}$ (Fig. 2e). Upon crossing the curves S_9 the separatrix value A_1 changes its sign from a positive to a negative one. The curve S_9 corresponds to the touching of $W^u_{O_2}$ with $W^s_{L_1}$.

Our qualitative and numerical analysis has revealed that:

1. In domains D_i $(i = 1, \ldots, 6)$ the system possesses only simple dynamics, either with one stable equilibrium state O_1 (domains D_1, D_4, D_6) or

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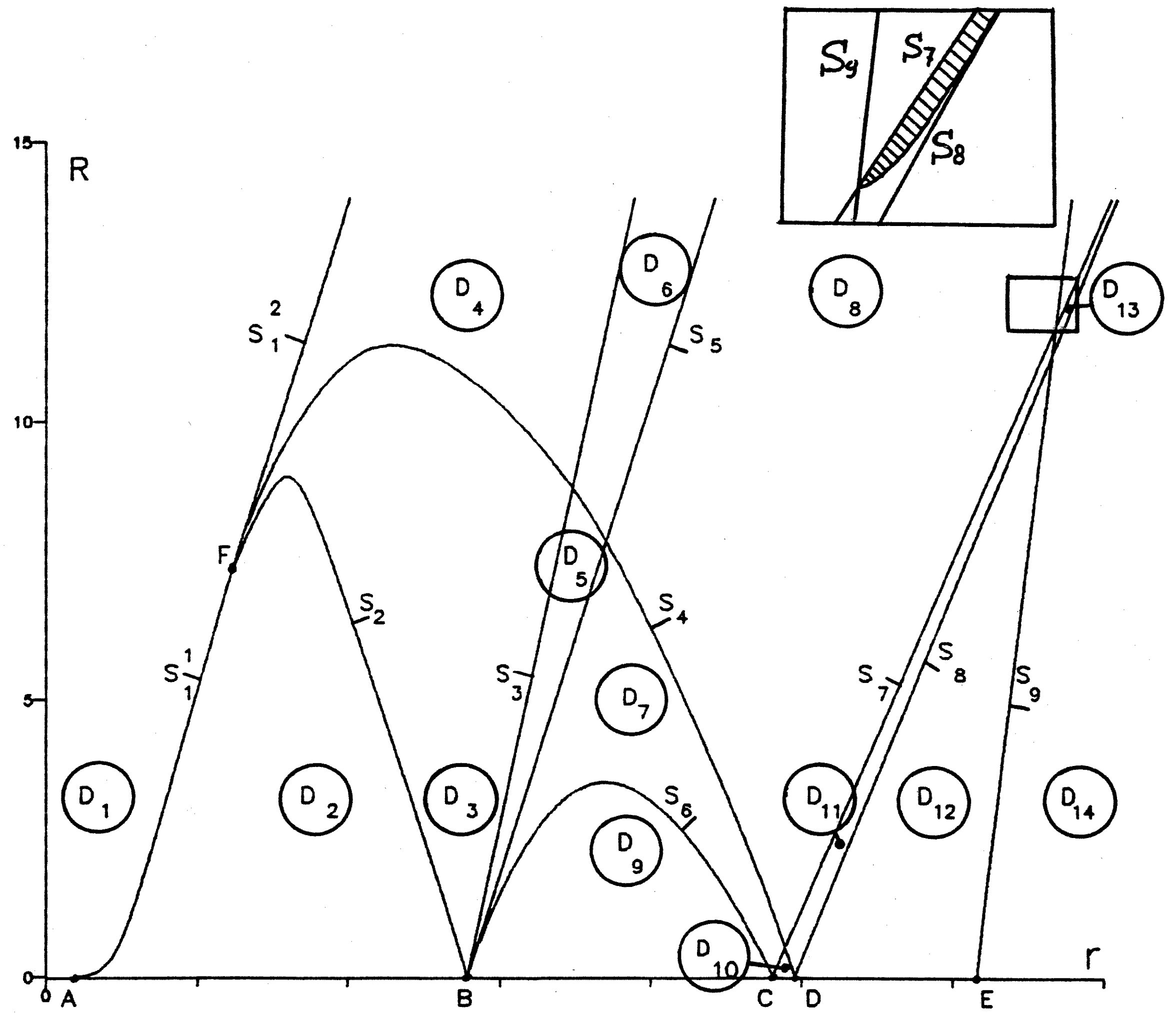

Fig. 1. The (r, R)-bifurcation diagram of Eq. (1) for $b = 8/3$ and $\sigma = 10$.

with two stable equilibrium states O_1, O_2 (domains D_2, D_3, D_5).

2. In domains D_i ($i = 7, 8, 9$) the system already displays a complex dynamics related to the existence of an unstable nontrivial one-dimensional limit set B^u. Note that in domains D_7 and D_9, there exists a continuum of bifurcation curves. It is related to the kneading nature of B^u. In the domains D_7 and D_9, the attractors are O_1 and O_2; in the domain D_8, the attractor is O_1.
3. In domains D_i ($i = 10, 11, 12$) the system possesses the Lorenz attractor; the attractors in the domain D_{10} are also O_1 and O_2; O_1 is the attractor in the domain D_{11}.
4. In domains D_{13} and D_{14} the system possesses a quasi-attractor, i.e. some attracting limit set containing a nontrivial hyperbolic set *and* stable periodic motions of a rather long period. In the domain D_{13} there exists a stable equilibrium state O_1 besides the quasi-attractor. Note that the point $E(30.98, 0.0)$ is a boundary for the existence of the Lorenz attractor in the symmetrical model.

The bifurcation diagram also contains a cross-hatched narrow domain. In this domain the system possesses an unstable nontrivial invariant set "within" which, generally speaking, there exist long-period stable periodic motions.

Below we describe the bifurcation values of r for $R = 3$ and 12, respectively.

The straight line $R = 3$ crosses the bifurcation curves S_i, $i = 1, \ldots, 9$: the curve S_1^1 at $r_1 = 3.837$, the curves S_2 and S_3 at $r_2 = 12.388$ and $r_3 = 15.340$ respectively, the curve S_5 at $r_5 = 15.743$, the curve S_6 at $r_6^1 = 16.328$ and $r_6^2 = 20.286$, the curve S_7 at $r_7 = 26.245$, the curve S_8

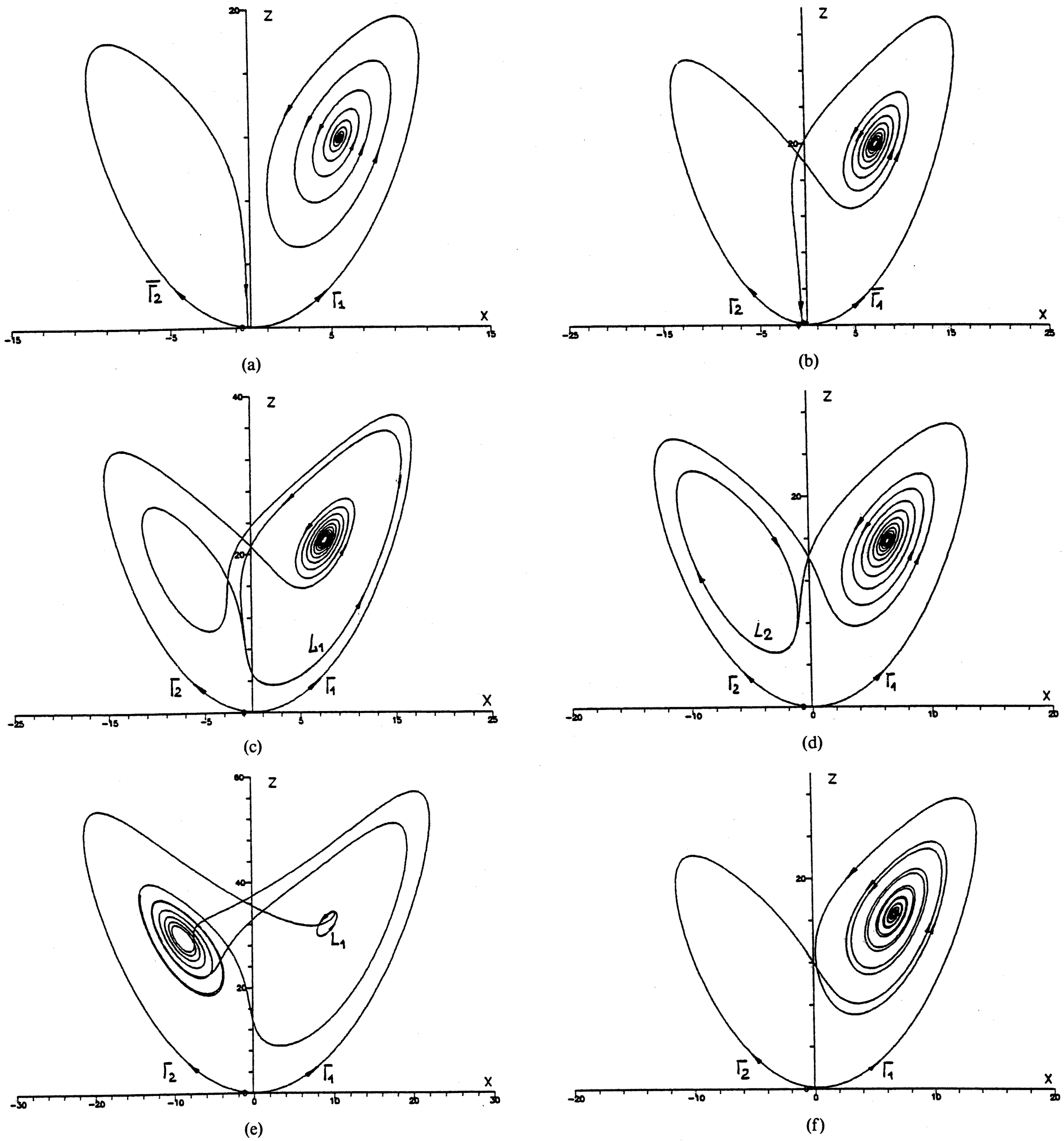

Fig. 2. Projections of the separatrices Γ_1 and Γ_2 in the X–Z plane for the following values of the parameters r and R: (a) $r = 12.388$, $R = 3.00$; (b) $r = 19.287$, $R = 12.00$; (c) $r = 21.290$, $R = 12.00$; (d) $r = 16.328$, $R = 3.00$; (e) $r = 31.752$, $R = 2.000$; (f) $r = 16.000$, $R = 12.00$.

at $r_8 = 26.721$, the curve S_4 at $r_4 = 16.529$, and the curve S_9 at $r_9 = 31.166$.

The straight line $R = 12$ crosses S_1^2 and S_i, $i = 3, 5, 7, 8, 9$, respectively, at $r_1 = 8.143$, $r_3 = 19.287$, $r_5 = 21.290$, $r_7 = 31.752$, $r_8 = 31.879$, and $r_9 = 31.454$. For $r_3 < r < r_7$ the system has only one stable equilibrium state O_1 (Fig. 2f). Note that $r_9 < r_7$, i.e., after crossing the curve S_7 to the right at $R = 12$, we do *not* have the Lorenz-type strange attractor but instead we have a quasi-attractor.

In conclusion we would like to note that the mathematical analysis of the complex dynamics of the system (1) is based upon the results obtained in Afraimovich *et al.* [1983], Afraimovich & Shil'nikov [1983], Afraimovich *et al.* [1987] and Bykov & Shil'nikov [1989].

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