ON THE NONSYMmetrical LORENZ MODEL*

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The structure of the bifurcation set for non-symmetric Lorenz model is studied. Regions of existence of simple and complex dynamics are pointed out. It is shown that there exist two qualitatively distinct scenarios of transition to chaos.

Consider the model

$$\begin{align*}
\dot{x} &= -\alpha(x-y), \\
\dot{y} &= -y+rx-xz+R, \\
\dot{z} &= -bz+yz
\end{align*}$$

(1)

which originates from the study of the liquid dynamics [Yorke & Yorke, 1981; Varandas & Mendonca, 1987] within a convective loop at the Lorenz parameters $b=\frac{8}{3}$ and $\alpha=10$. Since this model possesses the symmetry $(x, y, R) \leftrightarrow (-x, -y, -R)$, it is sufficient to study only for $R \geq 0$. In Fig. 1 a bifurcation diagram in the domain $0 \leq r \leq 32$, $0 \leq R \leq 13$ is presented. The points $A$, $B$, $C$, $D$, $E$ are well known and are located at the values $r_A = 1.0$, $r_B = 13.92$, $r_C = 24.06$, $r_D = 24.77$, and $r_E = 30.98$, respectively.

In the domain $D_1$ there exists one stable equilibrium state $O_1$. To the right of the curve $S^1 \cup F \cup S^2$ there are three equilibrium states $O_1$, $O_2$, and $O$. Crossing the curve $S^1$ the equilibrium states $O_2$ and $O$ are born from a saddle-node. Hence, $O_2$ is a stable equilibrium state and $O$ is a saddle of the type $(2, 1)$. Crossing the curve $S^2$ the equilibrium states $O_2$ and $O$ are born from an equilibrium state of the saddle-saddle type and so $O_2$ is a saddle of the type $(1, 2)$, and $O$ a saddle of the type $(2, 1)$. The point $F(6.077, 7.190)$ is a point of codimension two corresponding to a nonrough equilibrium state with two zero roots of the characteristic equation. This is a limit point for the two bifurcation curves $S_2$ and $S_4$. If by $\Gamma_1$ and $\Gamma_2$ we denote the trajectories starting from $O$ so that as $t$ tends to $\infty$ the trajectory $\Gamma_2$ tends to the equilibrium point $O$, then the curve $S_2$ will correspond to the homoclinic loop $\Gamma_2 = \Gamma_2 \cup O$ (Fig. 2a). On the curve $S_4$ the equilibrium state $O_2$ loses its stability. At this very instant the saddle periodic motion $L_2$ emerging from $\Gamma_2$ tightens into the equilibrium state $O$. The bifurcation curve $S_3$ corresponds to the homoclinic loop $\Gamma_1 = \Gamma_1 \cup O$ (Fig. 2b). Upon crossing the curve $S_3$ to the right the saddle periodic motion $L_1$ emerges from it. The curve $S_9$ is a stability boundary of equilibrium state $O_2$. On this boundary the motion $L_1$ tightens into the equilibrium state $O_1$. On the curve $S_9$, there is the inclusion $\Gamma_1 \in W^u_{l_1}$ (Fig. 2c). This curve separates the domain of simple dynamics (to the left) from the complex one (to the right). The curve $S_6$ corresponds to the inclusion $\Gamma_1 \in W^u_{l_2}$ (Fig. 2d) and the curve $S_7$ to the inclusion $\Gamma_2 \in W^u_{l_2}$ (Fig. 2e). Upon crossing the curves $S_9$ the separatrix value $A_1$ changes its sign from a positive to a negative one. The curve $S_9$ corresponds to the touching of $W^u_{O_2}$ with $W^s_{l_2}$.

Our qualitative and numerical analysis has revealed that:

1. In domains $D_i$ ($i = 1, \ldots, 6$) the system possesses only simple dynamics, either with one stable equilibrium state $O_1$ (domains $D_1, D_4, D_6$) or

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with two stable equilibrium states \( O_1, O_2 \) (domains \( D_2, D_3, D_5 \)).

2. In domains \( D_i \) (\( i = 7, 8, 9 \)) the system already displays a complex dynamics related to the existence of an unstable nontrivial one-dimensional limit set \( B^u \). Note that in domains \( D_7 \) and \( D_9 \), there exists a continuum of bifurcation curves. It is related to the kneading nature of \( B^u \). In the domains \( D_7 \) and \( D_9 \), the attractors are \( O_1 \) and \( O_2 \); in the domain \( D_8 \), the attractor is \( O_1 \).

3. In domains \( D_i \) (\( i = 10, 11, 12 \)) the system possesses the Lorenz attractor; the attractors in the domain \( D_{10} \) are also \( O_1 \) and \( O_2 \); \( O_1 \) is the attractor in the domain \( D_{11} \).

4. In domains \( D_{13} \) and \( D_{14} \) the system possesses a quasi-attractor, i.e. some attracting limit set containing a nontrivial hyperbolic set and stable periodic motions of a rather long period. In the domain \( D_{13} \) there exists a stable equilibrium state \( O_1 \) besides the quasi-attractor. Note that the point \( E(30.98, 0.0) \) is a boundary for the existence of the Lorenz attractor in the symmetrical model.

The bifurcation diagram also contains a cross-hatched narrow domain. In this domain the system possesses an unstable nontrivial invariant set “within” which, generally speaking, there exist long-period stable periodic motions.

Below we describe the bifurcation values of \( r \) for \( R = 3 \) and 12, respectively.

The straight line \( R = 3 \) crosses the bifurcation curves \( S_i, i = 1, \ldots, 9 \): the curve \( S_i^1 \) at \( r_1 = 3.837 \), the curves \( S_2 \) and \( S_3 \) at \( r_2 = 12.388 \) and \( r_3 = 15.340 \) respectively, the curve \( S_5 \) at \( r_5 = 15.743 \), the curve \( S_6 \) at \( r_6^1 = 16.328 \) and \( r_6^2 = 20.286 \), the curve \( S_7 \) at \( r_7 = 26.245 \), the curve \( S_8 \)
at $r_8 = 26.721$, the curve $S_4$ at $r_4 = 16.529$, and the curve $S_9$ at $r_9 = 31.166$.

The straight line $R = 12$ crosses $S_i^2$ and $S_i$, $i = 3, 5, 7, 8, 9$, respectively, at $r_1 = 8.143$, $r_3 = 19.287$, $r_5 = 21.290$, $r_7 = 31.752$, $r_8 = 31.879$, and $r_9 = 31.454$. For $r_3 < r < r_7$ the system has only one stable equilibrium state $O_1$ (Fig. 2f). Note that $r_9 < r_7$, i.e., after crossing the curve $S_7$ to the right at $R = 12$, we do not have the Lorenz-type strange attractor but instead we have a quasi-attractor.

In conclusion we would like to note that the mathematical analysis of the complex dynamics of the
system (1) is based upon the results obtained in Afraimovich et al. [1983], Afraimovich & Shil'nikov [1983], Afraimovich et al. [1987] and Bykov & Shil'nikov [1989].

References