"There is nothing more practical than a good theory."
James C. Maxwell

"... le souci du beau nous conduit aux mêmes choix que celui de l'utile."
Henri Poincaré

Preface

Many phenomena in science and technology are dynamical in nature. Stationary regimes, periodic motions and beats from modulations have long been believed to be the only possible observable states. However, discoveries in the second half of the 20th century have dramatically changed our traditional view of the character of dynamical processes. The breakthrough came with the discovery of a new type of oscillations called *dynamical chaos*. A deepening of our understanding of dynamical phenomena has since led us to a clear recognition that ours is a *nonlinear* world. This has resulted in the emergence of *nonlinear dynamics* as a scientific discipline whose aim is to study the common laws (regularities) of nonlinear dynamical processes.

A typical scheme for investigating a new phenomenon usually proceeds as follow: the relevant experiment or observation is studied by first constructing an adequate mathematical model in the form of dynamical equations. This model is analyzed and the result is compared with the experimental phenomenon.

This approach was first suggested by Newton. The laws that Newton discovered have provided a foundation for the mathematical modeling of numerous problems, including Celestial mechanics. The solution of the restricted two-body problem gives a brilliant explanation of the experimental Kepler's laws. In fact, starting with Newton, this method for modeling nature has dominated the field for many years. However, even such a purely scientific approach must be validated by questioning the correspondence between a real phenomenon and its phenomenological model, which had been aptly put by Brillouin: "A mathematical model differs from reality just as a globe differs from the earth".

A mathematical model in nonlinear dynamics usually consists of a system of equations with analytically given nonlinearities, and a finite number of parameters. The system may be described by ordinary differential equations, partial differential equations, equations with a delay, integro-differential equations, etc. In this book we will deal only with lumped (discrete-space) systems described by ordinary differential equations. Furthermore, we will restrict ourselves to a study of *non-conservative* systems thereby leaving aside the "ideal" dynamics of Hamiltonian systems (which Klein, at the end of the 19th century, had characterized as being the most "attractive mechanics without friction").

A system of differential equations is written in the form

$$\frac{dx}{dt} = X(x)\,,$$

where the independent variable t is called the time. One of the postulates of nonlinear dynamics which dates back to Aristotle and is based on common sense is that all observable states must be stable. This implies that in any comprehensive study of systems of differential equations, our attention must be focused on the character of the solutions over an *infinite* time interval. The systems considered from this point of view are called *dynamical*. Although the notion of a dynamical system is a mathematical abstraction — indeed we know from cosmology that even our Universe has only a *finite* life time — nevertheless, many phenomena of the real world have been successfully explained *via* the theory of dynamical systems. In the language of this theory the mathematical image of a stationary state is an equilibrium state, that of self-oscillations is a limit cycle, that of modulation is an invariant torus with a quasi-periodic trajectory, and the image of dynamical chaos is a strange attractor; namely, an attracting limit set composed of unstable trajectories.

In principle, the first three types of motions cited may be explained by a linear theory. That was the approach of the 19th century, which concerned mainly various practical applications modeled in terms of linear ordinary or partial differential equations. The most famous example is the problem of controlling steam engines whose investigation had led to the solution of the problem of stability of equilibrium states; namely to the classic Routh–Hurwitz criterion.

The most remarkable events in nonlinear dynamics can be traced to the twenties and the thirties of the 20th century. This period is characterized by the rapid development of radio-engineering. A common feature of many nonlinear radio-engineering problems is that the associated transient processes are typically very fast, thereby making it less time-consuming to carry out complicated experiments. The fact that the associated mathematical models in those days are usually simple systems of quasi-linear equations also plays an important role. This has in turn allowed researchers to conduct rather complete investigations of the models using methods based on Poincaré's theory of limit cycles and Lyapunov's stability theory.

Another significant event from that period is the creation of a mathematical theory of oscillations in two-dimensional systems. In particular, Andronov and Pontryagin identified a large class of rough (structurally stable) systems which admit a rather simple mathematical description. Moreover, all principal bifurcations of limit cycles were studied (Andronov, Leontovich) and complete topological invariants for both rough systems (Andronov, Pontryagin) and generic systems (Leontovich, Mayer) were described. Shortly after that, specialists from various areas of research applied these mathematically transparent and geometrically comprehensive methods to investigate concrete two-dimensional systems. This stage of the development is documented in the classic treatise "Theory of oscillations" by Andronov, Vitt and Khaikin.[1]

Further development in this subject included the attempt at a straightforward generalization of the concepts of planar systems, namely, the aim of extending the conditions of structural stability and bifurcations to the high-dimensional case. In no way does this approach indicate narrow visions. On the contrary, this was a mathematically sound strategy. Indeed, it was understood that entrance into space must bring new types of motions which may become crucial in nonlinear dynamics. As was mentioned previously, the mathematical image of modulation is a torus with quasi-periodic trajectories. Quasi-periodic trajectories are a particular case of almost-periodic trajectories which, by definition, are unclosed trajectories whose main feature is that they have *almost-periods* — the time intervals over which the trajectory returns close to its initial state. The quasi- and almost-periodic trajectories are self-limiting. A broader class of self-limiting trajectories consists of Poisson-stable trajectories. This kind of trajectory was discovered by Poincaré while studying the stability of the restricted three-body problem. A Poisson-stable trajectory also returns arbitrarily close to its initial state, but for an arbitrary but fixed small neighborhood of the initial state, the sequence of the associated return

[1]This book was first published in 1937 but without the name of Vitt, who had already been repressed.

times may be unbounded, *i.e.* the motion is unpredictable. In accordance with Birkhoff's classification, stationary, periodic, quasi-periodic, almost-periodic and Poisson-stable trajectories exhaust all types of motions associated with non-transient behaviors.

In the early thirties Andronov posed the following basic question in connection with the mathematical theory of oscillations: Can a Poisson-stable trajectory be Lyapunov stable? The answer was given by Markov: If a Poisson-stable trajectory is stable in the sense of Lyapunov (to be more precise, uniformly stable), then it must be almost-periodic. It seemed therefore that no other motions, apart from those which are almost-periodic, exist in nonlinear dynamics. Therefore, despite new discoveries in the qualitative theory of high-dimensional systems in the sixties it was not clear whether this theory had any value beyond pure mathematics. But this did not last long.

For within a relatively short period of time Smale had established the foundation for a theory of structurally stable systems with complex behavior in the trajectories, a theory that is generally referred to nowadays as *the hyperbolic theory.* In essence, a new mathematical discipline with its own terminology, notions and problems has been created. Its achievements have led to one of the most amazing fundamental discoveries of the 20th century — *dynamical chaos.*[2] Hyperbolic theory had provided examples of strange attractors which might be the mathematical image of chaotic oscillations, such as the well-known turbulent flows in hydrodynamics.

Nevertheless, the significance of strange attractors in nonlinear dynamics were not widely appreciated, especially not by specialists in turbulence. There were a few reasons for their reluctance. By mathematical construction, known hyperbolic attractors possess such a complex topologically structure that it did not allow one to conceive of any reasonable scenarios for their emergence. This has led one to regard hyperbolic attractors as being the result of a pure abstract scheme irrelevant to real dynamical processes.[3] Moreover, the phenomenon of *chaos* which has been observed in many concrete models could scarcely be associated with hyperbolic attractors because of the appearance of stable periodic orbits of long periods, either for the given parameter values, or for nearby ones. This enabled skeptics to argue that any observable chaotic

[2]Chronologically, this discovery came after the creation of "relativity theory" and "quantum mechanics".

[3]The possibility of applying hyperbolic attractors to nonlinear dynamics remains problematic even today.

behavior represents a transient process only. In this regard, we must emphasize that the persistence of the unstable behavior of trajectories of a strange attractor with respect to sufficiently small changes in control parameters is the essence of the problem: In order for a phenomenon to be observable it must be stable with respect to external perturbations.

The breakthrough in this controversy came in the mid seventies with the appearance of a simple low-order model

$$\dot{x} = -\sigma(x - y)\,,$$
$$\dot{y} = rx - y - xz\,,$$
$$\dot{z} = -bz + xy\,,$$

where chaotic behavior in its solutions was discovered numerically by E. Lorenz in 1962. A detailed analysis carried out by mathematicians revealed the existence of a strange attractor which is not hyperbolic but structurally unstable. Nevertheless, the main feature persisted, namely, the attractor preserved the instability behavior of the trajectories under small smooth perturbations of the system. Such attractors, which contain a single equilibrium state of the saddle type, are called *Lorenz attractors.* The second remarkable fact related to these attractors is that the Lorenz attractor may be generated via a finite number of easily observable bifurcations from systems endowed with only trivial dynamics.

Since then, dynamical chaos has been almost universally accepted as a legitimate and fundamental phenomenon of nature. The Lorenz model has since become a *de facto* proof of the existence of chaos, even though the model itself, despite its hydrodynamical origin, contains “too little water”.[4] More recently, a much more realistic mathematical model of a real physical system called *Chua’s Circuit* has also been proved rigorously to exhibit dynamical chaos, and whose experimental results agree remarkably well with both mathematical analysis and computer simulations [76–79].

We will not discuss further the relevance of the theory of strange attractors but note only that the theory of nonlinear oscillations created in the thirties had been so clear and understandable that generations of nonlinear researchers were able to apply it successfully to solve problems from many scientific disciplines. A different situation occurred in the seventies. Limit cycles and tori

[4]The Lorenz system represents the simplest Galerkin approximation of the problem of the convection of a planar layer of fluid.

which exhibit a unified character were replaced by strange attractors which possess a much more complex mathematical structure. They include smooth or non-smooth surfaces and manifolds, sets with a local structure represented as a direct product of an interval and a Cantor set, or even more sophisticated sets. Today, a specialist in complex nonlinear dynamics must either have a strong mathematical background in the qualitative theory of high-dimensional dynamical systems, or at least a sufficiently deep understanding of its main statements and results. We wish to remark that just as nonlinear equations cannot usually be integrated by quadratures, the majority of concrete dynamical models do not admit "a qualitative integration" by a purely mathematical analysis. This inevitably leads to the use of computer analysis as well. Hence, an ultimate requirement for any formal statement in the qualitative theory of differential equations is that it must have a complete and concrete character. It must also be free of unnecessary restrictions which, paraphrasing Hadamard, are not dictated by the needs of science but by the abilities of the human mind.

In most cases, the parameter space of a high-dimensional model may be partitioned into two regions according to whether the model exhibits simple or complex behaviors in its trajectories. The primary indication or sign of the presence of complex behavior will be associated in this book with the presence of a Poincaré homoclinic trajectory. Although Poincaré had discovered these trajectories in the restricted three-body problem, *i.e.* in a Hamiltonian system, such trajectories are essential objects of study in all fields of nonlinear dynamics as well. In general, the presence of Poincaré homoclinic trajectories leads to rather important conclusions. It was simultaneously established by Smale and L. Shilnikov (from opposite locations on the globe) that systems with a Poincaré homoclinic trajectory possess infinitely many co-existing periodic trajectories and a continuum of Poisson-stable trajectories. All of them are unstable. In essence, these homoclinic structures are the elementary bricks of dynamical chaos.

As for high-dimensional systems with simple behavior of trajectories, they are quite similar to planar systems [80]. In principle, the only new feature is the possibility of the existence in the phase space of an invariant torus with a quasi-periodic trajectory covering the torus. So, any concrete model may be completely analyzed in this region of the parameter space.

The situation is fundamentally different in the case of systems with complex trajectory behavior. Indeed, it has been established recently by Gonchenko,

L. Shilnikov and Turaev that a complete analysis of most models of nonlinear dynamics is unrealistic [28].

This book is concerned only with the qualitative theory of high-dimensional systems of differential equation with simple dynamics. For an extremely rich variety of such systems which arise in practical applications, the reader is referred to the very large systems of nonlinear differential equations (typically with dimensions greater than 10,000 state variables) associated with *Cellular Neural Networks* [81], which include lattice dynamical systems and cellular automata as special cases. We have partitioned this book into two parts. The first part is mainly introductory and technical in nature. In it we consider the behavior of trajectories close to simple equilibrium states and periodic trajectories, as well as discuss some problems related to the existence of an invariant torus. It is quite natural that we first present the classical results concerning the stability problem. Of special concern are the unstable equilibria and periodic trajectories of the saddle type. Such trajectories play a crucial role in the contemporary qualitative theory. For example, saddle equilibrium states may form unseparated parts of strange attractors. Saddles are also related to some principally important problems of a nonlocal character, etc. Our technique for investigating the behavior of systems near saddle trajectories in this book is based on the method suggested by L. Shilnikov in the sixties. The main feature of this method is that the solution near a saddle is sought not as a solution of the Cauchy problem but as a solution of a special boundary-value problem. Since this method has not yet been clearly presented in the literature, but is known only to a small circle of specialists, it is discussed in detail in this book.

In the second part of this book we analyze the principal bifurcations of equilibrium states, as well as of periodic, homoclinic and heteroclinic trajectories. The theory of bifurcations has a key role in nonlinear dynamics. Its roots go back to the pioneering works of Poincaré and Lyapunov on the study of the form of a rotating fluid. A bifurcation theory based on the notion of roughness, or structural stability, has since been developed. Whereas in the rough (robust) case small changes do not induce significant changes in the states of a system, the bifurcation theory explains what happens in the non-rough case, including many possible qualitative transformations. Some of these transitions may be dangerous, possibly leading to catastrophic and irreversible situations. The bifurcation theory allows one to predict many real-world phenomena. In particular, notions such as the soft and the rigid (severe) regimes of excitation of oscillations, the safe and dangerous boundaries of the stability regions of steady states and periodic motions, hysteresis, phase-locking, etc., have all been formulated and analyzed via bifurcation theory.

In this book we give special attention to the boundaries of stability of equilibria and periodic trajectories in the parameter space. Along with standard bifurcations, both local and global, we also examine a bifurcation phenomenon discovered recently by L. Shilnikov and Turaev [66], the so-called "blue sky catastrophe". The essence of this phenomenon is that in the parameter space there may exist stability boundaries of a periodic trajectory such that upon approaching the boundary both the length and the period of the periodic trajectory tend to infinity, whereas the periodic orbit resides at a finite distance from any equilibrium state in a bounded region of the phase space. This bifurcation has not yet been observed in models of physical systems, although a three-dimensional two-parameter model with a polynomial right-hand side is known [25].

This book is essentially self-contained. All necessary facts are supplied with complete proofs except for some well-known classical results such as the Poincaré–Denjoy theory on the behavior of trajectories on an invariant torus.

The basis of this book is a special course in which the first author gave at the Nizhny Novgorod (formerly, Gorky) University over the last thirty years. This course usually proceeds with a one-year lecture on the qualitative theory of two-dimensional systems, which was delivered by Prof. E. A. Leontovich-Andronova for many years. Besides that, discussions on certain aspects of this course had formed the subject of student seminars, and weekly scientific seminars at the Department of Differential Equations of the Institute for Applied Mathematics & Cybernetics. This book will appeal to beginners who have chosen the qualitative theory and the theory of bifurcations and strange attractors as their majors. Undoubtedly, this book will also be useful for specialists in the above subjects and in related mathematical disciplines, as well as for a broad audience of interdisciplinary researchers on nonlinear dynamics and chaos, who are interested in the analysis of concrete dynamical systems.

Part I of this book consists of six chapters and two appendices.

In Chap. 1 we describe the principal properties of an autonomous system, give the notion of an abstract dynamical system and select the principal types of trajectories and invariant sets necessary for further presentation. In addition, we discuss some problems of qualitative integration of differential equations which is based on the notion of topological equivalence. The material of this chapter also has reference value, beginners may call on it when needed.

In Chap. 2 we examine the behavior of trajectories in a neighborhood of a structurally-stable equilibrium state. Our approach here goes back to Poincaré. Using this approach we classify the main types of equilibrium states. Special attention is given to equilibria of the saddle types, and, in particular, to leading and nonleading (strongly stable) invariant manifolds. We also give sufficient attention to the asymptotic representation of solutions near a saddle point. As mentioned, our methods are based on Shilnikov's boundary-value problem. In addition, we prove some theorems on invariant manifolds. We would like to stress that along with well-known theorems on stable and unstable manifolds of a saddle, some rather important results which we will need later are given here. In the last section of the chapter some useful information concerning Poincare's theory of resonances for local bifurcation problems are presented.

In Chap. 3 we discuss structurally-stable periodic trajectories. Our consideration is focused on the behavior of trajectories of the Poincaré map in a neighborhood of the fixed point. As in the case of equilibria we investigate an associated boundary-value problem near a saddle fixed point and prove a theorem on the existence of its invariant manifolds. Sections 3.10–3.12 and 3.14 are concerned only with the properties of periodic trajectories in continuous time.

Invariant tori are considered in Chap. 4. More specifically, we study a non-autonomous system which depends periodically, as well as quasi-periodically, on time. This class of non-autonomous system can be extended to higher dimensions by adding some equations having a specific form with respect to cyclic variables. To prove the existence of an invariant torus in such a system, we use a universal criterion, the so-called annulus principle which is applicable for systems with small perturbations. In the case of a periodic external force, the behavior of the trajectories on a two-dimensional invariant torus may be modeled by an orientable diffeomorphism of a circle. In relation to this we present a brief review of some related results from the Poincaré–Denjoy theory. We complete this chapter with a discussion of an important problem of nonlinear dynamics, namely, the synchronization problem associated with the phenomenon of "beats" in modulations.

The final two chapters, Chap. 5 and 6, are dedicated to local and global center manifolds, respectively. We re-prove in Chap. 5 a well-known result that in a small neighborhood of a structurally unstable equilibrium state, or near a bifurcating periodic trajectory of a $\mathbb{C}^r$-smooth dynamical system, there exists locally an invariant $\mathbb{C}^r$-smooth center manifold whose dimension is equal to the number of characteristic exponents with a zero real part in the case of equilibrium states, or to the number of multipliers lying on a unit circle in the case of periodic trajectories. Our proof of the center manifold theorem relates it to the study of a specific boundary value problem and covers all basic local invariant manifolds (strongly stable and unstable, extended stable and unstable, and strongly stable and unstable invariant foliations). We discuss how the existence of the center manifold and the invariant foliation allows one to reduce the problem of investigating the local bifurcations of a system to that of a corresponding sub-system on the center manifold, thereby significantly decreasing the dimension of the problem.

In Chap. 6 the proof of the analog of the theorem on the center manifold for the case of global bifurcations is presented. Unlike the local case, the dimension of the non-local center manifold does not depend on the degree of degeneracy of the Jacobian matrix, but is equal to some integer which can be estimated in terms of the numbers of negative and positive characteristic exponents of saddle trajectories comprising a heteroclinic cycle. Another characteristic of the non-local center manifold is that it is only $\mathbb{C}^1$-smooth in general. The restriction on such center manifolds may only be used for studying those bifurcation problems which admit the solution within the framework of $\mathbb{C}^1$-smoothness. Therefore, in contrast to the local bifurcation theory, one cannot directly apply non-local center manifolds to study various delicate bifurcation phenomena which require more smoothness. Hence, the theorem contains, in essence, certain qualitative results which only allow us to anticipate some possible dynamics of the trajectories in a small neighborhood of a homoclinic cycle, as well as to estimate the dimensions of the stable and unstable manifolds of trajectories lying in its neighborhood, and, consequently, to evaluate the number of positive and negative Lyapunov exponents of these trajectories. We consider in detail only the class of systems possessing the simplest cycle; namely, a bi-asymptotic trajectory (a homoclinic loop) which begins and ends at the same saddle equilibrium state. We then extend this result to general heteroclinic cycles.

In the Appendix we prove a theorem on the reduction of a system to a special form which is quite suitable for analysis of the trajectories near a saddle point. This theorem is especially important because an often postulated assumption on a straight-forward linearization of the system near a saddle may sometimes lead to subsequent confusion when more subtle details of the behavior of the trajectories are desired. The essence of our proof is a technique (based on the reduction of the problem to a theorem on strong stable invariant manifold) for making a series of coordinate transformations which are robust to small, smooth perturbations of the system. We will use this special form in the second part of this book when we study homoclinic bifurcations.

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