# INTRODUCTION TO PART II

In the following chapters we present the theory of bifurcations of dynamical systems with simple dynamics. It is difficult to over-emphasize the role of bifurcation theory in nonlinear dynamics the reason is quite simple: the methods of the theory of bifurcations comprise a working tool kit for the study of dynamical models. Besides, bifurcation theory provides a universal language to communicate and exchange ideas for researchers from different scientific fields, and to understand each other in interdisciplinary discussions.

Bifurcation theory studies the changes in the phase space as we vary the parameters of the system. In essence, this is the authentic notion of bifurcation theory proposed originally by Henry Poincaré when he studied Hamiltonian systems with one degree of freedom. We must, however, note that this intuitively evident definition is not always sufficient at the contemporary stage of the development of the theory. One needs, in fact, to have an appropriate mathematical foundation to define the notions of the structure of the phase space and the changes in the structure.

The first attempt at creating such formalization had been made by Andronov and Pontryagin in 1937: they introduced the notion of a *rough* system. For a system to be rough, it means that any sufficiently close system is to be topologically equivalent to the given one. Moreover, the conjugating homeomorphism must be close to identity. In other words, the two systems must have matching phase portraits and corresponding trajectories can differ only slightly.

In the same paper, Andronov and Pontryagin had presented the necessary and sufficient conditions of roughness for systems on the plane. Consequently, many problems of nonlinear dynamics that can be modeled by two-dimensional dynamical systems has since attained a necessary mathematical foundation.

The main statements of the Andronov and Pontryagin theory are presented in the first section of Chap. 7, which opens Part II of this book. We also give the definition of structural stability (due to Peixoto) there. The difference between the notion of structural stability and that of roughness is that, the conjugating diffeomorphism defining the structural stability is not assumed to be close to identity in the former case. This is rather convenient from a purely mathematical point of view as it follows immediately from the definition that structurally stable systems form an open set. Even though numerous known proofs had only concentrated on structural stability, roughness itself follows from the same proofs as a by-product. Hence, the difference of these two notions does not seem to be that essential. Note, nonetheless, that the notion of structural stability has become much more widely known outside of Russia, especially in the West. In this book we will frequently utilize this term as well. In spite of that, we believe that the notion of roughness is, in principle, more reasonable as it gives the natural image of small changes of real processes caused by small variations of parameters.

The multi-dimensional extension of two-dimensional rough systems is the Morse–Smale systems discussed in Sec. 7.4. The list of limit sets of such a system includes equilibrium states and periodic orbits only; furthermore, such systems may only have a finite number of them. Morse–Smale systems do not admit homoclinic trajectories. Homoclinic loops to equilibrium states may not exist here because they are non-rough — the intersection of the stable and unstable invariant manifolds of an equilibrium state along a homoclinic loop cannot be transverse. Rough Poincaré homoclinic orbits (homoclinics to periodic orbits) may not exist either because they imply the existence of infinitely many periodic orbits. The Morse–Smale systems have properties similar to two-dimensional ones, and it was presumed (before and in the early sixties) that they are dense in the space of all smooth dynamical systems. The discovery of dynamical chaos destroyed this idealistic picture.

The fundamental question of "what distinguishes systems with simple dynamics from systems with chaotic dynamics?" can only be answered if we can correspond certain types of trajectories to physically observable processes. We began the classification with the study of quasiperiodic trajectories (Chap. 4 in the first part of this book). Even though these trajectories are non-rough, they were shown to model adequately such phenomena as beats and modulations.

Quasiperiodic trajectories are a special case of Poisson-stable trajectories. The latter plays one of the leading roles in the theory of dynamical systems as they form a large class of center motions in the sense of Birkhoff (Sec. 7.2). Birkhoff had partitioned the Poisson-stable trajectories into a number of subclasses. This classification is schematically presented in Sec. 7.3. Having chosen this scheme as his base, as early as in the thirties, Andronov had undertaken an attempt to collect and correlate all known types of dynamical motions with those observable from physical experiments. Since his arguments were based on the notion of stability in the sense of Lyapunov for an individual trajectory, Andronov had soon come to the conclusion that all possible Lyapunov-stable trajectories are exhausted by equilibrium states, periodic orbits and almost-periodic trajectories (these are quasiperiodic and limit-quasiperiodic motions in the finite-dimensional case).

Thus, it appeared naturally to assume that every interesting dynamical regime possesses a discrete frequency spectrum. In this connection, it is curious to note that Landau and Hopf had proposed quasiperiodic motions with a sufficiently large number of independent frequencies as the mathematical image of hydrodynamical turbulence (the number of the frequencies was supposed to increase to infinity as some structural parameter, such as the Reynolds number, increases).

All other Poisson-stable trajectories are unstable in the sense of Lyapunov. How can such trajectories be of any use in dynamics? The answer was found nearly 30 years later. For the first time, the significance of a stable limit set consisting of individually unstable trajectories for explaining the complex and chaotic behavior of nonlinear dynamical processes was recognized by Lorenz in 1963 [87].

In the rough case an analysis of the structure of such a limit set (called a quasiminimal set, which is defined as the closure of an unclosed Poisson-stable trajectory) may be performed using Pugh's closing lemma. The main conclusion that follows from this analysis (see Sec. 7.3) is that periodic orbits are dense in a rough quasiminimal set. In particular, we will see that the number of periodic orbits is infinite. Systems possessing such limit sets are called systems with *complex dynamics*.

A more vivid characteristics of systems with complex behaviors is the presence of a Poincaré homoclinic trajectory, i.e. a trajectory which is biasymptotic to a saddle periodic orbit as $t \to \pm\infty$. The existence of a homoclinic orbit which lies at the transverse intersection of the stable and unstable invariant manifolds of the saddle periodic orbit implies the existence of infinitely many other saddle periodic orbits in the phase space (Sec. 7.5).

However, rough systems (both types — with simple and complex dynamics) with dimension (of the phase space) greater than two are not dense in the space of dynamical systems. In fact, it turns out that a key role must have been given to *non-rough* attracting limit sets with unstable behaviors in their trajectories.

An example of such a set is the Lorenz attractor which occurs in a variety of models. The wild spiral attractor [153] is another fascinating example.[1]

The similarity between both strange attractors is that none contains stable periodic orbits. The difference between them is that all Poincaré homoclinic orbits in the Lorenz attractor are rough, whereas the featuring property of the wild attractor is the coexistence of rough and non-rough Poincaré homoclinic orbits due to homoclinic tangencies. The similarity is that both attractors are "concentrated" on a rough equilibria state which is a saddle in the case of the Lorenz attractor, and a saddle-focus in the case of the wild attractor. Among other features of models with such strange attractors, we may single out the existence of regions in the parameter space where the parameter values corresponding to homoclinic loops to the equilibrium state are dense.

A complete understanding of such complex phenomena is impossible without a thorough knowledge of basic bifurcations, both local and global. General aspects of this theory are reviewed in Chap. 8. We begin the analysis with the simplest non-rough systems in the two-dimensional case, following the pioneering works by Andronov and Leontovich. They carried out a systematic classification of all principal bifurcations of limit cycles on the plane of which there are four sub-types: namely, the birth of a limit cycle from:

(1) a simple weak focus;
(2) a simple semistable limit cycle;
(3) a separatrix loop to a simple saddle-node; and
(4) a separatrix loop to a saddle at which the divergence of the vector field is non-zero.

The Andronov–Leontovich classification employs an additional notion of the so-called degree of non-roughness. A further development of the theory

[1] The spiral-like shape of this attractor follows from the shape of homoclinic loops to a saddle-focus (2, 1) which appear to form its skeleton. Its wildness is due to the simultaneous existence of saddle periodic orbits of different topological type and both rough and non-rough Poincaré homoclinic orbits.

had taken yet another direction, namely by selecting bifurcation sets of codimension one for primary bifurcations, and of arbitrary (though finite) codimension in the general case. Moreover, even though all two-dimensional flows on a connected component of a bifurcation surface of a given finite codimension are all topologically equivalent (Leontovich–Mayer theorem), this is no longer true in the multi-dimensional case.

This result is due to Palis, who had found that two-dimensional diffeomorphisms with a heteroclinic orbit at whose points an unstable manifold of one saddle fixed point has a quadratic tangency with a stable manifold of another saddle fixed point can be topologically conjugated locally only if the values of some continuous invariants coincide. These continuous invariants are called moduli. Some other non-rough examples where moduli of topological conjugacy arise are presented in Sec. 8.3.

Surprisingly, even non-rough systems of codimension one may have infinitely many moduli. Of course, since the models of nonlinear dynamics are explicitly defined dynamical systems with a *finite* set of parameters, this creates a new obstacle which the classical bifurcation theory has not run into. Although the case of homoclinic loops of codimension one does not introduce any principal problem, nevertheless codimensions two and higher are much less trivial as, for example, in the case of a homoclinic or heteroclinic cycle including a saddle-focus where the structure of the bifurcation diagrams is directly determined by the specific values of the corresponding moduli.

Therefore, Andronov's approach (Sec. 8.4) for studying dynamical models has to be corrected in cases where a complete bifurcation analysis may not be possible without moduli. We note, however, that if some fine delicate phenomena may be ignored, or if the problem is restricted to the analysis of non-wandering orbits like equilibrium states, periodic and quasiperiodic motions, a study of the main bifurcations in systems with simple dynamics still remains realistic within the framework of finite-parameter families under certain reasonable requirements (Sec. 8.4).

We note parenthetically that the situation becomes drastically different for the systems with complex dynamics. In the majority of cases (at least in those cases where homoclinic tangencies appear) the introduction of the moduli is inexorable because they serve as the essential parameters governing the bifurcations (see [63]).

Although the theory of the typical bifurcations of limit cycles in two-dimensional systems was created by Andronov and Leontovich in the thirties,[2] a systematic development of the bifurcation theory of periodic orbits and equilibrium states in multi-dimensional systems was initiated only after their results became available to the scientific community (the work of Hopf in 1942 was, perhaps, the only exception).

A straightforward generalization of two-dimensional bifurcations was developed soon after. So were some natural modifications such as, for instance, the bifurcation of a two-dimensional invariant torus from a periodic orbit. Also it became evident that the bifurcation of a homoclinic loop in high-dimensional space does not always lead to the birth of only a periodic orbit. A question which remained open for a long time was: could there be other codimension-one bifurcations of periodic orbits? Only one new bifurcation has so far been discovered recently in connection with the so-called "blue-sky catastrophe" as found in [152]. All these high-dimensional bifurcations are presented in detail in Part II of this book.

In Chaps. 9 and 10 we consider structurally unstable equilibrium states and periodic orbits. The bifurcations of these limit sets are studied in Chap. 11. These three chapters belong to a theory of *local* bifurcations. The results with local bifurcations are well presented in the literature and this theory continues to develop rapidly. We therefore restrict ourselves here to a detailed study of the basic cases. First of all, for a bifurcating equilibrium state whose characteristic exponents do not lie on the imaginary axis, we assume that they lie strictly to the left of it. On the imaginary axis we assume that there is either a single zero exponent,[3] or a complex-conjugate pair of pure imaginary ones. Analogous assumptions are made in the case of periodic motions: the multipliers which do not lie on the unit circle must lie inside it, and those on the unit circle consist of a single multiplier equal to $+1$, or $-1$, or a complex-conjugate pair $e^{\pm i\varphi}$, $0 < \varphi < \pi$. The corresponding bifurcations in these cases are sufficiently simple, so wherever it is possible we do not impose restrictions on the nonlinear terms.

The reason for our assumption on the spectrum of characteristic exponents is quite obvious: we focus special attention on the problem of the loss of stability of equilibrium states and periodic motions and on the bifurcations accompanying the loss of stability. It is clear that these problems are a primary subject of nonlinear dynamics.

[2]This was reported in the preface of the first edition of the book "The Theory of Oscillations" by Andronov, Vitt and Khaikin (which was printed without the name of Vitt in 1937).

[3]The case of a double-zero characteristic exponent is partly considered in Sec. 13.2.

Of course, the cases of higher degeneracies in the linear part are also very interesting; for example, an equilibrium state with three characteristic exponents 0, $\pm i\omega$, or with two pairs of purely imaginary exponents $\pm i\omega_1$, $\pm i\omega_2$, etc. In such cases of codimension two it is typical that the associated (truncated) normal form reduces to a two-dimensional system with a finite number of parameters. A systematic study of these normal forms is presented in [21, 40, 64, 82].

One must bear in mind, however, that a truncated normal form does not always guarantee a complete reconstruction of the dynamics of the original system. For instance, when the truncated normal forms possess additional symmetries, these symmetries are, in principle, broken if the omitted higher-order terms are taken back into account, and this can even lead to an onset of chaos in some regions of the parameter space. These regions are extremely narrow near a bifurcation point of codimension two but their size may expand rapidly as we move away from the bifurcation point over a finite distance.

The significance of higher degeneracies (starting from codimension three) in the linear part is that the effective normal forms become three-dimensional, and may, as a result, exhibit complex dynamics, the so-called instant chaos, even in the normal form itself. Such examples include the normal forms for a bifurcation of an equilibrium state with a triplet of zero characteristic exponents, and a complete or incomplete Jordan block, in which there may be a spiral strange attractor [18], or a Lorenz attractor [129], respectively (the latter case requires an additional symmetry). Since we will focus our considerations only on simple dynamics, we do not include these topics in this book.

The key methods in our presentation of local bifurcations are based on the center manifold theorem and on the invariant foliation technique (see Sec. 5.1. of Part I). The assumption that there are no characteristic exponents to the right of the imaginary axis (or no multipliers outside the unit circle) allows us to conduct a smooth reduction of the system to a very convenient "standard form." We use this reduction throughout this book both in the study of local bifurcations on the stability boundaries themselves and in the study of global bifurcations on the route over the stability boundaries (Chap. 12).[4] These

[4] In the general case where there are both stable and unstable characteristic exponents, or stable and unstable multipliers in the spectrum, the local bifurcation problem does not cause any special difficulties, thanks to the reduction onto the center manifold. Consequently, the pictures from Chaps. 9–11 will need only some slight modifications where unstable directions replace stable ones, or be added to existing directions in the space. However, the reader must

global bifurcations are related to the fact that in contrast to an equilibrium state which always persists on any boundary of its stability region, a periodic orbit may not exist on the stability boundary. In particular, a periodic orbit may disappear via one of the following scenarios:

(1) it shrinks to an equilibrium state;
(2) a saddle-node equilibrium state appears suddenly on it;
(3) it adheres into a homoclinic loop to a saddle equilibrium state; and
(4) it undergoes a blue-sky catastrophe, when its period and length both become infinite when it approaches a stability boundary. In contrast to homoclinic bifurcations, no equilibrium state is involved in a blue-sky catastrophe.

In Chap. 12 we will study the global bifurcations of the disappearance of saddle-node equilibrium states and periodic orbits. First, we present a multi-dimensional analogue of a theorem by Andronov and Leontovich on the birth of a stable limit cycle from the separatrix loop of a saddle-node on the plane. Compared with the original proof in [130], our proof is drastically simplified due to the use of the invariant foliation technique. We also consider the case when a homoclinic loop to the saddle-node equilibrium enters the edge of the node region (non-transverse case).

The bifurcation of a separatrix loop of a saddle-node was discovered by Andronov and Vitt [14] in their study of the transition phenomena from synchronization to beating modulations in radio-engineering. Specifically, they had studied the periodically forced van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x = \mu A \sin \omega t \,,$$

where $\mu \ll 1$ and $\omega_0 - \omega \sim \mu$. In the associated averaged equation, they showed the existence of the saddle-node bifurcation which explained the simple transition from a stable equilibrium state to a periodic motion. However, the question of the correspondence between the limit sets of the averaged equation and those of the original one was not solved then. Andronov and Vitt returned to this problem in their succeeding paper [15] where, using the method of a small parameter by Poincaré, they proved the correspondence between the rough equilibrium state of an averaged system and a periodic orbit

be aware that since a reduction to the standard form is not always smooth in this general case, it cannot be applied in a straightforward way to the analysis of certain global bifurcations (such as the disappearance of saddle-saddle equilibria or saddle-saddle periodic orbits).

of the original system. Later on, Krylov and Bogolyubov [81] proved the correspondence between the rough periodic orbit in the averaged equations and the two-dimensional invariant torus in the original system. Thus, a rigorous explanation of the transition from synchronization to modulations in the original system requires a study of the bifurcation of the possible birth of an invariant torus at the disappearance of a saddle-node periodic orbit.

The general setting of the problem of global bifurcations on the disappearance of a saddle-node periodic orbit is as follows. Assume that there exists a saddle-node periodic orbit and that all trajectories which tend to this periodic orbit as $t \to -\infty$ also tend to it as $t \to +\infty$ along some center manifold. In other words, assume that the unstable manifold $W^u$ of the saddle-node returns to the saddle-node orbit from the side of the node region. In this case, either:

(1) $W^u$ is a two-dimensional invariant manifold such as a torus, or a Klein bottle, or
(2) $W^u$ is not a manifold.

If the system has a global cross-section (which always exists when we treat a periodically forced autonomous system), the unstable manifold $W^u$ will only be a torus. The intersection of $W^u$ with the cross-section is a closed curve which is invariant under the Poincaré map. Consequently, the following two cases are possible:

(1) the curve is smooth, and
(2) the curve is non-smooth.

If the curve is smooth when the saddle-node disappears, a closed attracting invariant curve remains on the cross-section. This result is due to Afraimovich and Shilnikov [3]. If the invariant curve is non-smooth, the situation becomes essentially more complicated, because the disappearance of the saddle-node may now lead the original system out of the Morse–Smale class, i.e. the system may exhibit complex structures. Afraimovich and Shilnikov discovered if the so-called "big lobe" or "small lobe" conditions are satisfied, then there exists a sequence of parameter intervals corresponding to the occurrence of complex dynamics. This result was subsequently improved by Newhouse, Palis and Takens [97] who proved that there exists a sequence of parameter values corresponding to a transverse homoclinic orbit (and, hence, there always exists a sequence of intervals corresponding to complex dynamics), without using the big lobe condition but restricted to one-parameter families of a special kind. An analogous result for this bifurcation for general one-parameter families is also obtained in [151] where it is shown that if the big lobe condition is satisfied, then chaos exists for all (small) parameter values just after a saddle-node's disappearance. On the contrary, if this condition is not satisfied, then intervals of complex dynamics and those exhibiting only simple dynamics (a continuous invariant curve exists) must alternate on the parameter axis.

Note that the effect of alternating zones of simple and complex behavior was discovered for the first time by van der Pol [154] in his experiments on the periodic forcing of a lamp generator (this effect occurs when one tunes a radio, and a characteristic noise is heard while moving from one station to another). The first theoretical explanation was given by Cartwright and Littlewood [36] for the van der Pol equation.

We will present in Sec. 12.2 a summary of results for the case where the unstable manifold $W^u$ of the saddle-node is homeomorphic to a torus along with the proof of a theorem on the persistence of the invariant torus in the smooth case. There, we will also develop a general theory for an effective reduction of the problem to a study of some family of endomorphisms (smooth non-invertible maps) of a circle.

When a system does not have a global cross-section, the unstable manifold $W^u$ of the saddle-node may also be a Klein bottle (if the system is defined in $R^n$ with $n \geq 4$). If the Klein bottle is smooth at the bifurcation point, it will persist after the disappearance of the saddle-node. For topological reasons, a pair of periodic orbits will always exist on the Klein bottle such that the length of both orbits will increase to infinity while approaching the event of the sudden appearance of the original saddle-node. Generically, these periodic orbits will change stability infinitely many times via a forward and backward period-doubling bifurcations. If the Klein bottle is non-smooth at the bifurcation point, then the big lobe or the small lobe conditions should be applied. The former guarantees complex dynamics for all small values of the parameter beyond the demise of saddle-node. In contrast, the small lobe condition can only guarantee the existence of a sequence of intervals of parameter values where complex dynamics occurs. Note that unlike the case where $W^u$ is homeomorphic to a torus, in the case of a non-smooth Klein bottle the dynamics may be simple for all small parameter values when the small lobe condition is not satisfied (the case of a "very small lobe"). These results are presented in Sec. 12.3.

A totally different situation becomes possible in the case where the system does not have a global cross-section, and when $W^u$ is not a manifold. In this case (Sec. 12.4), the disappearance of the saddle-node periodic orbit may, under some additional conditions, give birth to another (unique and stable) periodic orbit. When this periodic orbit approaches the stability boundary, both its length and period increases to infinity. This phenomenon is called *a blue-sky catastrophe*. Since no physical model is presently known for which this bifurcation occurs, we illustrate it by a number of natural examples.

Note that in the $n$-dimensional case, where $n \geq 4$, other topological configurations of $W^u$ may be realized. Such saddle-node bifurcations will definitely lead the system out of the class of systems with simple dynamics. For example, it is shown in [139, 152] that a hyperbolic attractor of the Smale–Williams type may appear just after the disappearance of a saddle-node periodic orbit.[5]

Another typical codimension-one bifurcation (left untouched in this book) within the class of Morse–Smale systems includes the so-called saddle-saddle bifurcations, where a non-rough saddle equilibrium state with one zero characteristic exponent (the others lie in both left and right half-planes) coalesces with another saddle having a different topological type. If, in addition, the stable and unstable manifolds of the saddle-saddle point intersect each other transversely along some homoclinic orbits, then as the bifurcating point disappears, saddle periodic orbits are born from the homoclinic loops. If there is only one homoclinic loop, then only one periodic orbit is born from it, and respectively, this bifurcation does not lead the system out of the Morse–Smale class. However, if there are more than one homoclinic loops, a hyperbolic limit set with infinitely many saddle periodic orbits will appear after the saddle-saddle vanishes [135].

A similar effect occurs when a saddle-saddle periodic orbit (with one multiplier equal to 1 and the rest of the multipliers both inside and outside of the unit circle) disappears. If the stable and unstable manifolds of the saddle-saddle periodic orbits intersect across two (at least) smooth tori, then the disappearance of such a periodic orbit is followed by the birth of a limit set in which an infinite set of smooth saddle invariant tori is dense [6].

[5] A more general case is also considered in [139] concerning the disappearance of a saddle-node torus and followed by the appearance of Anosov attractors and multi-dimensional solenoids.

In Chap. 13 we will consider the bifurcations of a homoclinic loop to a saddle equilibrium state. We start with the two-dimensional case. First of all, we investigate the question of the stability of the separatrix loop[6] in the generic case (non-zero saddle value), as well as in the case of a zero saddle value. Next, we elaborate on the cases of arbitrarily finite codimensions where the so-called Dulac sequence is constructed, which allows one to determine the stability of the loop via the sign of the first non-zero term in this sequence.

In the case of a non-zero saddle value, we present the classical result by Andronov and Leontovich on the birth of a unique limit cycle at the bifurcation of the separatrix loop. Our proof differs from the original proof in [9] where Andronov and Leontovich essentially used the topology of the plane. However, following Andronov and Leontovich we present our proof under a minimal smoothness requirement ($\mathbb{C}^1$).

The case of zero saddle value was considered by E. A. Leontovich in 1951. Her main result is presented in Sec. 13.3, rephrased in somewhat different terms: in the case of codimension $n$ (i.e. when exactly the first $(n-1)$ terms in the Dulac sequence are zero) not more than $n$ limit cycles can bifurcate from a separatrix loop on the plane; moreover, this estimate is sharp.

In the same section we give the bifurcation diagrams for the codimension two case with a first zero saddle value and a non-zero first separatrix value (the second term of the Dulac sequence) at the bifurcation point. Leontovich's method is based on the construction of a Poincaré map, which allows one to consider homoclinic loops on non-orientable two-dimensional surfaces as well, where a small-neighborhood of the separatrix loop may be a Möbius band. Here, we discuss the bifurcation diagrams for both cases.

The bifurcations of periodic orbits from a homoclinic loop of a multi-dimensional saddle equilibrium state are considered in Sec. 13.4. First, the conditions for the birth of a stable periodic orbit are found. These conditions stipulate that the unstable manifold of the equilibrium state must be one-dimensional and the saddle value must be negative. In fact, the precise theorem (Theorem 13.6) is a direct generalization of the Andronov–Leontovich theorem to the multi-dimensional case. We emphasize again that in comparison with the original proof due to Shilnikov [130], our proof here requires only the $\mathbb{C}^1$-smoothness of the vector field.

[6]Only one-sided stability is naturally considered.

We consider next the homoclinic bifurcation of the saddle whose unstable manifold is still one-dimensional, but the saddle value is now assumed to be positive. Unlike the case of the negative saddle value, here we need some additional non-degeneracy conditions to be imposed on the system. These conditions, in fact, imply the existence of a stable two-dimensional invariant $\mathbb{C}^1$-manifold in the system, which is either a cylinder or a Möbius band, depending on the sign of the so-called separatrix value. Hence, our problem is reduced, essentially, to the two-dimensional case considered in Sec. 13.2. Since this problem is a particular case of a more general problem (the case of the multi-dimensional unstable manifold) considered in Sec. 13.5, we focus more on the geometry underlying the result. Such an approach is relevant to the study of the Lorenz attractors, as well as some other homoclinic bifurcations of higher codimensions.

We end this section with a consideration of the homoclinic loop to a saddle-focus whose unstable manifold is one-dimensional. It is shown that when the saddle value is positive, infinitely many saddle periodic orbits coexist near such a homoclinic loop of the saddle-focus (Theorem 13.8).

The existence of complex dynamics near a homoclinic loop to a saddle-focus was discovered by L. Shilnikov for the three-dimensional case in [131]. Subsequently, the four-dimensional case[7] was considered in [132]; and the general case in [136].

In Sec. 13.5 we consider the bifurcation of the homoclinic loop of a saddle without any restrictions on the dimensions of its stable and unstable manifolds. We prove a theorem which gives the conditions for the birth of a single periodic orbit from the loop [134], and also formulate (without proof) a theorem on complex dynamics in a neighborhood of a homoclinic loop to a saddle-focus. Here, we show how the non-local center manifold theorem (Chap. 6 of Part I) can be used for simple saddles to reduce our analysis to known results (Theorem 13.6).

In the case of the saddle-focus, the result of [136] in its full generality cannot be obtained by a reduction to any invariant manifold. However, generically (i.e. under some simple non-degeneracy conditions) the problem can be reduced to a three- or four-dimensional invariant manifold [120, 150].

Section 13.6 discusses three main cases of codimension-two bifurcations of a homoclinic loop to a saddle. These cases were selected by Shilnikov in [138]

[7]Here, the saddle-focus has two pairs of complex-conjugate characteristic exponents and the divergence of the vector field is non-vanishing at the saddle-focus.

for explaining the immediate onset of the Lorenz attractor from a homoclinic butterfly. Later, these bifurcations attracted much interest (see references in Sec. 13.6). Here we consider a multi-dimensional case of a homoclinic loop to a saddle with zero saddle value and those cases of the so-called "orbit-flip" and "inclination-flip" bifurcations which do not lead to complex dynamics. Although the corresponding bifurcation diagrams are widely known (see [126, 77, 129] for the inclination-flip case, [119] for the orbit-flip case, and [99, 38, 77, 65] for the case of zero saddle-value), an explicit and complete proof is published here, probably for the first time.

In Sec. 13.7 we describe two other cases of codimension two, namely the bifurcations of a homoclinic-8 and a heteroclinic cycle with two saddles. Both cases are considered within the Morse–Smale class (we require the saddle-value to be negative in the case of the homoclinic-8; in the case of the heteroclinic cycle, either the saddle values must be negative or the conditions which guarantee the existence of a two-dimensional invariant manifold must be satisfied). The results surveyed in this section are extracted from [148, 151, 50, 149] for the homoclinic-8, and [121, 122, 123, 124, 125] for the heteroclinic cycles. Some other results on heteroclinic connections with a different topology [34, 35] are also presented. The structure of bifurcation diagrams in the case where two saddle-foci are involved is much more complicated in contrast to the case of the connection between two saddles (even though the dynamics remains simple in both cases). According to [158], the fine structure of the bifurcation diagrams for the saddle-focus case is sensitive to arbitrarily small changes of the continuous topological invariants (moduli) discussed in Sec. 8.3.

The last chapter focuses on the general problems of the transition over the stability boundaries of equilibrium states and periodic orbits. These questions have an immediate significance for the subject of nonlinear dynamics, specially in cases where changes in the parameters of a working device may push it out of its stability region, or when the control parameters are deliberately chosen as close to the stability boundary as possible in order to achieve maximal performance. For stationary regimes, the corresponding problems were addressed by Bautin in his monograph first published in 1949. He classified stability boundaries as either safe or dangerous. When a safe boundary is crossed, the representative phase point does not leave a small neighborhood of the bifurcating equilibrium state or periodic orbit, although the latter becomes unstable. In the case of a dangerous boundary, the phase point blows out from a small neighborhood of the bifurcating trajectory. Evidently, a local analysis becomes inadequate in the case of dangerous boundaries: one must investigate here how the unstable sets behave at the critical moment. For instance, if a stable limit cycle adheres to a homoclinic loop of a saddle, it becomes crucial to know where the other separatrix goes to since its $\omega$-limit set will be the new dynamical regime of the system. In other cases, it turns out, however, that there may be more than one stable limit set included in the boundary of the unstable set at the critical parameter value (if this bifurcation is within the Morse–Smale class, these limit sets are stable equilibria or periodic orbits). Another option embraces the so-called dynamically indefinite stability boundary where a random choice of the new regime occurs as a natural dynamical phenomenon — the dynamical uncertainty.

The number of papers and monographs on the theory of bifurcations is very large and increasing rapidly. Some of the questions considered in this book are, to a certain extent, reflected in other books as well (see especially the books marked by an asterisk in the list of references). We stress, however, that in many works, while studying global bifurcations, the assumption of smooth linearization of the equations near equilibrium states and periodic orbits is very often made only for the sake of maximal convenience. The linearization assumption requires the absence of resonances, which in turn imposes an infinite set of unnecessary additional conditions on the system (or, the number of such assumptions, first finite, may grow very fast as the dimension of the system grows). Therefore, any approach based on linearization will cast some doubts on the full applicability of the theoretical results to dynamical models.[8] The methods presented in this book are free from these problems. This is achieved by the use of techniques developed by our research group in Nizhny Novgorod. It is applied in Chaps. 12 and 13 to non-local bifurcations. We stress that we need only a very small degree of smoothness. This, perhaps, makes our analysis more complicated, but it guarantees and enhances the validity and the adequacy of our global bifurcation results. The methods presented in this book are applicable also for systems with complex dynamics, in particular, for systems with homoclinic tangencies [58, 59, 62], see also [100, 101].

[8]It happens rather often that some results which sound fine mathematically, being formulated for "typical" or "generic" families of dynamical systems, when applied to a specific problem require the verification of their stipulated conditions. It is unfair, however, to force a researcher to consume time and computational resources only to check on conditions which are, in fact, unnecessary.

We would like to acknowledge the help given to us while working on this book. In particular, we wish to thank S. Gonchenko, M. Shashkov, O. Stenkin, L. Lerman and J. Moiola. We would also like to acknowledge the generous supports from the USA Office of Naval Research and the Swiss Federal Institute of Technology (EPFL and ETH).