Selecta Mathematica Sovietica
Vol. 10, No. 2 (1991)

0272-9903/91/020105-13 $1.50 + 0.20/0

Bifurcation and Chaos in the Marioka–Shimizu System*

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This paper is concerned with the qualitative analysis of the limit regimes and strange attractors (SA) in the Marioka–Shimizu model [**6**]

$$\dot{x} = y, \qquad \dot{y} = -xz + x - \lambda y, \qquad \dot{z} = x^2 - \alpha z, \qquad \alpha, \lambda > 0, \tag{1}$$

where α and λ are positive parameters.

This system was proposed as an alternative model for studying the dynamics of the Lorenz equations for large Rayleigh numbers (Ra). Actually, the Lorenz model

$$\dot{x}_c = -\sigma(x_c - y_c), \qquad \dot{y}_c = -x_c z_c + \mathrm{Ra}\, x_c - y_c, \qquad \dot{z}_c = x_0 y_c - b z_c \tag{2}$$

with the substitutions

$$x_H = \frac{\varepsilon}{\sqrt{2\sigma b}} x_c, \qquad y_H = \frac{\varepsilon^2}{\sqrt{\sigma}}(y_c - x_c), \qquad z_H = \varepsilon^2\left(z_c - \frac{x_c^2}{2\sigma}\right), \qquad t_H = \frac{\sqrt{\sigma'}}{\varepsilon} t_c$$

and the introduction of new parameters

$$\alpha \triangleq \varepsilon b \sigma^{-1/2}, \qquad \beta \triangleq \varepsilon(2\sigma - b)^{-1/2}, \qquad \lambda \triangleq \varepsilon(\sigma + 1)\sigma^{-1/2}, \qquad \varepsilon \triangleq (\mathrm{Ra} - 1)^{-1/2}$$

is transformed [**4**] to the form

$$\dot{x}_H = y_H, \qquad \dot{y}_H = -\frac{x_H^3}{\beta} - x_H z_H + x_H - \lambda y_H, \qquad \dot{z}_H = x_H^2 - \alpha z_H. \tag{3}$$

System (1) is obtained from (3) in the limit case for $\beta \to \infty$. In this system, stable symmetric and asymmetric periodic motions, as well as stochastic behavior

* Originally published in Methods of the Qualitative Theory of Differential Equations, Gorky State University, 1986, pp. 180–193. Translated by Paul B. Slater.

of trajectories, were discovered by the authors of [**6**] through a computer calculation.

As in the Lorenz model, the Marioka–Shimizu system is invariant with respect to the substitution $(x, y, z) \to (-x, -y, z)$. The system (1) has three equilibrium states: $O(0, 0, 0)$ and $O_{1,2}(\pm\alpha^{1/2}, 0, 1)$. It is easy to see that the roots of the characteristic equation are the following:

$$k_1 = -\lambda/2 - (\lambda^2/4 + 1)^{1/2}, \qquad k_2 = -\alpha, \qquad k_3 = -\lambda/2 + (\lambda^2/4 + 1)^{1/2},$$

that is, O is a saddle of type (2.1). In $O_{1,2}$ the characteristic equation has the form

$$\gamma^3 + (\alpha + \lambda)\gamma^2 + \alpha\lambda\gamma + 2\alpha = 0.$$

The boundary of the domain of stability is determined by the relation $R = (\alpha + \lambda)\lambda - 2 = 0$. For $R > 0$, $O_{1,2}$ are the stable foci, and for $R < 0$, $O_{1,2}$ are of saddle-focus type (1.2). On the stability boundary $R = 0$, the characteristic equation has the roots

$$\gamma_{1,2} = \pm i(2 - \lambda^2), \qquad \gamma_3 = -2/\lambda.$$

As we know, the question of the nature of the bifurcation as one crosses $R = 0$ is decided by the sign of the first Lyapunov value. To calculate it, let us use the method worked out by N. N. Bautin [**3**]. Following [**3**], [**7**], let us reduce system (1) to a single third-order equation

$$\dddot{x} + (\alpha + \lambda)\ddot{x} + \alpha\lambda\dot{x} - \alpha x + x^3 - \frac{\lambda}{x}\dot{x}^2 - \frac{\dot{x}\ddot{x}}{x} = 0.$$

Let us move the origin in the equilibrium state by substituting $x = \xi + x_0$, where $x_0 = \alpha^{1/2}$. After the substitution, the necessary terms for computing the first Lyapunov value will be taken into account if, in the expansion, $(\xi + x_0)^{-1}$ is bounded by terms of degree no higher than the first. As a result, we will obtain the equation

$$\dddot{\xi} + (\alpha + \lambda)\ddot{\xi} + \alpha\lambda\dot{\xi} - (\alpha - 3x_0^2)\xi + 3x_0\xi^2 - \xi^3 - \frac{\lambda}{x_0}\dot{\xi}^2 + \frac{\lambda}{x_0^2}\xi\dot{\xi}^2 - \frac{\dot{\xi}\ddot{\xi}}{x_0} + \frac{\xi\dot{\xi}\ddot{\xi}}{x_0^2} = 0,$$

which we write anew in the form of the system

$$\dot{\xi}_1 = -(\alpha + \lambda)\xi_1 - \lambda\alpha\xi_2 - 2\alpha\xi_3 - 3x_0\xi_3^2 + \frac{\xi_1\xi_2}{x_0} + \xi_3^3 + \frac{\lambda}{x_0}\xi_2^2 - \frac{\lambda}{x_0^2}\xi_3\xi_2^2 - \frac{\xi_1\xi_2\xi_3}{x_0^2},$$

$$\dot{\xi}_2 = \xi_1, \qquad \dot{\xi}_3 = \xi_2.$$

In our case, the expression for the first Lyapunov value, calculated with the boundary $R = 0$, looks like the following:

$$\mathscr{L}_1 = \frac{\pi p^2 q^{3/2}}{8\Delta_0^2(p^2+4q)}(-3\lambda^{12}+23\lambda^{10}-\lambda^9-53\lambda^8+4\lambda^7$$
$$+18\pi^6+\lambda^5+48\lambda^4-10\lambda^3+16\lambda^2-4\lambda-64),$$
$$\Delta_0 = q^{3/2}(p^2+q), \qquad p = 2/\lambda, \qquad q = 2-\lambda^2.$$

The determination of the sign of $\mathscr{L}_1$ was conducted on the computer by the Sturm method, with the help of the system of analytic computations REDUCE. It turned out that on the curve of neutrality $R = 0$: $\alpha = (2-\lambda^2)/\lambda$, $\lambda \in]0, \sqrt{2}]$, $\mathscr{L}_1$ is negative. Consequently, as one passes across the stability boundary $R = 0$, stable periodic motions S_1 and S_2 are generated as a result of a direct Andronov–Hopf bifurcation from the equilibrium states O_1 and O_2. In this manner, in contrast to the Lorenz model, in the Marioka–Shimizu system the stability boundary of the nonzero equilibrium states is secure.

As also in the Lorenz model, an important bifurcated curve is l_1 (see Figure 1)—the formation of simple homoclinic curves $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ of the saddle O. However, in our case, the saddle variable $\sigma = -\alpha - \lambda/2 + (\lambda^2/4+1)^{1/2}$ on l_1 can

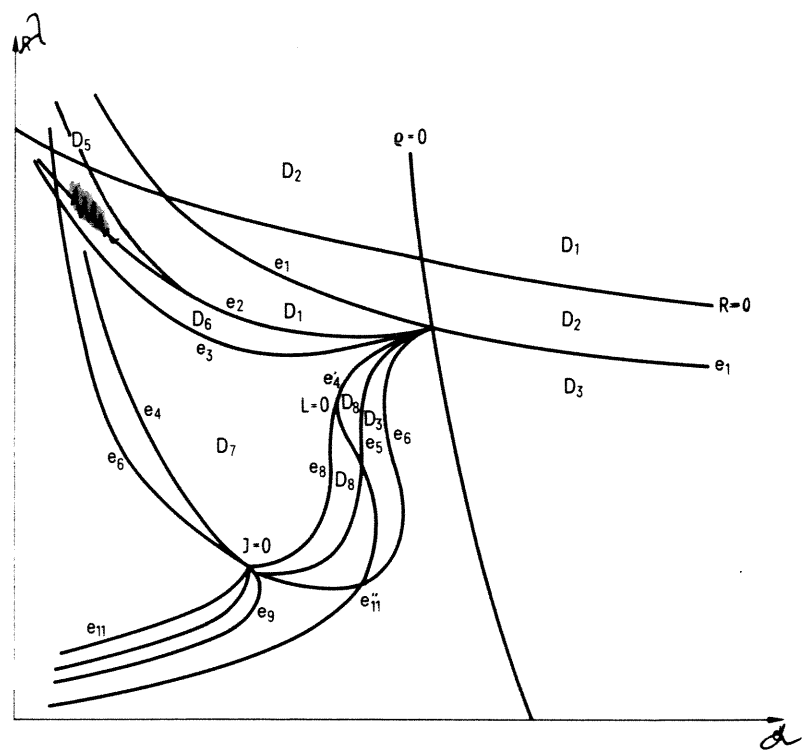

Figure 1.

assume both negative and positive values. Therefore, the further study of the system naturally breaks into two cases.

The case $\sigma < 0$. Here, everything is simple. As one passes the curve $R = 0$ from the domain D_1 to D_2 (see Figure 1), stable periodic motions S_1 and S_2 are generated from O_1 and O_2. On l_1, the motions S_1 and S_2 go respectively, into $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. With the destruction of the contour $\bar{\Gamma} = \overline{\Gamma_1 \cup O \cup \Gamma_2}$ in D_3, a stable symmetric $(x, y) \to (-x, -y)$ periodic motion M appears. Let us note that in D_3, we have that M is the unique stable limiting set.

The case $\sigma > 0$. All the necessary conditions of the theorem from [**8**] are satisfied on l_1: the separatrices Γ_1 and Γ_2 return to O along the leading direction (the z-axis) and the separatrix value A is positive. Therefore, as one crosses l_1 from D_1 to D_4, from each $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ there are generated, according to the saddle periodic motion, C_1 and C_2, respectively. On l_3, we have that C_1 and S_1, and C_2 and S_2, merge, forming structurally unstable periodic motions of saddle-node type, and disappear. As is known [**1**], along with C_1 and C_2, there is also generated an invariant Ω-limit set, homeomorphic to the superstructure over the Bernoulli scheme with two symbols, possessing a countable set of saddle periodic motions, everywhere dense in it. This set is not attractive and, consequently, the stable sets will be O_1 and O_2 in D_4, and S_1 and S_2 in D_5.

The bifurcation curve l_2 corresponds to the birth of an SA of the Lorenz attractor type. This moment is characterized by the fact that on l_2 the separatrices Γ_1 and Γ_2 lie on the two-dimensional stable manifolds W^s of C_2 and C_1. Thus, as in the Marioka–Shimizu model, there occurs something like the beginning of an SA, similar to the Lorenz model [**1**]. A numerical analysis of the system revealed the following peculiarity: at the point $\sigma = 0$ on l_1, the first separatrix value A is positive and less than unity. Therefore, the formation of a lacuna with a saddle symmetric periodic motion G (see below) can precede the disappearance of the SA.

The point $\sigma = 0$ is the boundary of the following bifurcation curves (see Figure 2):

l_2—the origin of an SA of the Lorenz attractor type;

l_3—single-circuit asymmetric periodic motions of saddle-node type;

l_4—a symmetric periodic motion by a unit multiplier;

l_5—the boundary of the domain of existence of the SA;

l_6—double-circuit or 0-1 type (in relation to those occurring on l_1) homoclinic curves of the saddle O.

On l_6 there exists a point at which the separatrix value A vanishes. This point breaks l_6 into two homoclinic curves of orientable and nonorientable type, respectively. On l_4 there is a point where the second Lyapunov value $\mathscr{L}$ of the

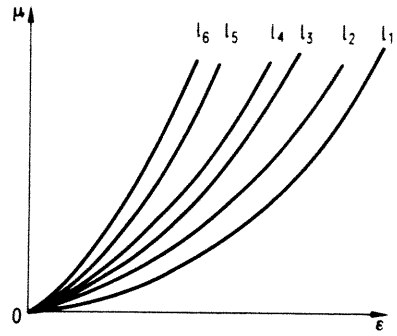

Figure 2.

Poincaré mapping of an area element, transversal to the symmetric periodic motion, equals zero. The first Lyapunov value equals zero, by the symmetry of the system.

The point $A = 0$ serves as the boundary for the curves l_5 and l_7, determining the domain of the SA. Also, l_8, the curve of the origin of two structurally unstable double-circuit asymmetric periodic motions of saddle-node type, joins this point. The second boundary l_8 is the point $\mathscr{L} = 0$ on l_4. This point divides l_4 into two: l_4', where $\mathscr{L} > 0$, and l_4'', where $\mathscr{L} < 0$.

As was already noted, the domain of existence of the SA is bounded by l_5 and l_7 besides l_2. Let us consider the evolution of the SA as one passes from D_7 to D_3 across l_5 and l_4' in the first case, and l_5 and l_4'', in the second.

In the first case, on l_4' ($\mathscr{L} > 0$) the symmetric saddle periodic motion G disintegrates into three, and in D_3 there exist in the lacuna three periodic motions: a stable M and a pair of asymmetric periodic motions of saddle type, k_1 and k_2. The bifurcation on l_5 is similar to that occurring on l_2, with the only difference that Γ_1 and Γ_2 lie on the W^s of k_1 and k_2. In D_9, the separatrices Γ_1 and Γ_2 now approach M. On l_6 Γ_1 and Γ_2 become doubly-asymptotic to O, forming a double-circuit homoclinic curve, into which k_1 and k_2 go.

In the second case, let us describe the bifurcations as one passes across l_5 and l_4'' in the reverse order, that is, from D_3 to D_7. As already noted, M is the unique stable limit set in D_3 to which Γ_1 and Γ_2 approach. As one passes across l_6 from the homoclinic curves of the saddle O, k_1 and k_2 are born. On l_4'' ($\mathscr{L} < 0$) the cycle M bifurcates into three: G and two stable asymmetric periodic motions N_1 and N_2. Now Γ_1 and Γ_2 approach N_1 and N_2. On l_5 Γ_1 and Γ_2 lie on the W^s of k_1 and k_2. Thus, there exist five periodic motions in the lacuna: G, $N_{1,2}$ and $k_{1,2}$. The last merge pairwise on l_8, forming a structurally unstable periodic motion of saddle-node type, and disappear. It is apparent that, in this case, the chain of bifurcations is similar to the mechanism of the origin of a SA in crossing l_2, with

the only difference being that the role of the single-circuit asymmetric periodic motions is played by the two-circuit asymmetric ones.[1]

There is one more boundary of the domain of existence of a SA of Lorenz attractor type, namely l_7, on which the separatrix value A vanishes. Crossing l_7 into the domain where $A < 0$ leads to having the SA of Lorenz type become a quasi-attractor. The cause of this is the appearance of homoclinic Poincaré curves, and consequently, of stable periodic motions, since the divergence of the vector field is negative. As was shown in [**2**], their existence is connected with the formation of Smale horseshoes. The latter also means that while crossing l_7, saddle periodic motions with negative multipliers appear.

As was noted above, the point $A = 0$ on l_6 serves as the boundary of l_7. In a neighborhood of this point, the curves l_9, l_{10} and l_{11}, corresponding to the bifurcations of the doubling of the period, originate. As one crosses l_9, the cycles N_1 and N_2 lose stability, and cycles of a doubled period appear, etc. In all, six steps were computed in the cascade of doublings. In analogy with the Feigenbaum mechanism, a magnitude $\delta \simeq 4.0$ was found.

The structure of the bifurcation set in a neighborhood of $\sigma = 0$

As follows from [**9**], the nature of the behavior of the system in the neighborhood of such a point essentially depends on the sign and value of the separatrix value A. It was shown in [**10**] that the basic bifurcation curves (the formations of the lacuna, the loops of the separatrices, . . .) of a two-dimensional Poincaré mapping on a transversal area element in a sufficiently small neighborhood W_0^s of the saddle O (depending on σ) can be fully studied with the help of a one-dimensional mapping of the form

$$T_\mu : \bar{x} = (-\mu + Ax^{1-\varepsilon}) \operatorname{sign}(x), \tag{4}$$

where μ is a parameter corresponding to the coordinates of the first intersection of the separatrices with the section $\varepsilon = \sigma/|k_2|$, where k_2 is the least negative eigenvalue in O. The value $\mu = 0$ corresponds in (1) to the formation of simple homoclinic curves O on l_1, and $\mu < 0$ corresponds to the domain of the parameters located above l_1. We will consider the case $A < 1$, since, as a numerical analysis of system (1) showed, the limit set on the curve $\sigma = 0$ is the stable symmetric periodic motion M. The refined value $A = 0.81$ was obtained with the help of the Poincaré map at the point $\sigma = 0$ on l_1.[2]

Let us consider the case $\varepsilon \lesssim 0$. For $\mu < 0$ (see Figure 3) the mapping possesses two stable fixed points S_1 and S_2. As $\mu \to 0$, S_1 and S_2 go into the point of

[1] This mechanism of the death of a SA was "phenomenologically" described in [**1**].

[2] Let us observe that a similar mapping with $A > 1$ was also considered in [**11**].

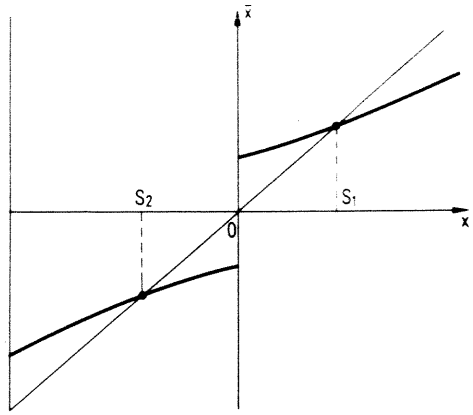

Figure 3.

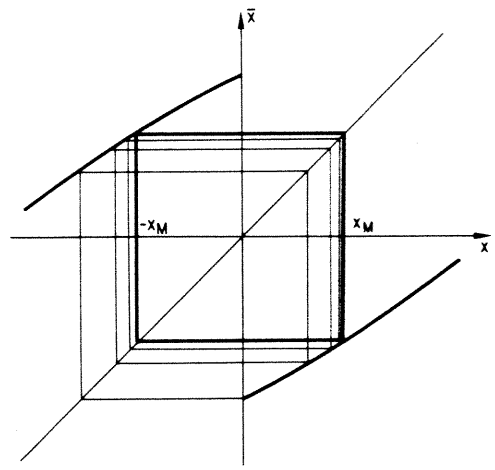

Figure 4.

discontinuity. For $\mu < 0$, from the discontinuity point, a stable cycle M of period two is generated (see Figure 4), one of the coordinates of which, by symmetry, is found as the fixed point of the mapping, of the form (see Figure 5)

$$\bar{x} = -(-\mu + Ax^{1-\varepsilon}).$$

The case $\varepsilon \gtrsim 0$

For $\mu < 0$, as in the first case, the mapping (4) has only the stable points S_1 and S_2 (see Figure 6). As one passes across $\mu = 0$ from the point of discontinuity,

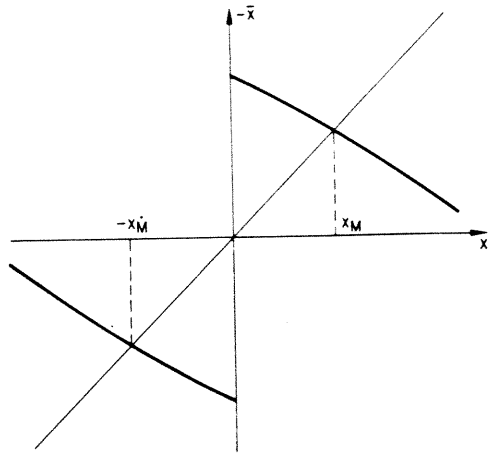

Figure 5.

there appear two unstable points C_1 and C_2 along with the unstable Ω_1 limit set, homeomorphic to a one-sided Bernoulli scheme with two symbols, the extreme points of which are C_1 and C_2 (see Figure 7).

The curve l_1 (see Figure 2) is the boundary of the origin of the Lorenz attractor. The moment of bifurcation ensues when (see Figure 8)

$$x_{C_1} = \mu, \qquad x_{C_1} = -\mu + A x_{C_1}^{1-\varepsilon}.$$

Hence we obtain $l_1 : \mu = (A/2)^{1/\varepsilon}$.

As μ increases, the points S_1 and C_1 (S_2 and C_2) draw together, and on l_2 they form a nonrough point of saddle-node type (see Figure 9).

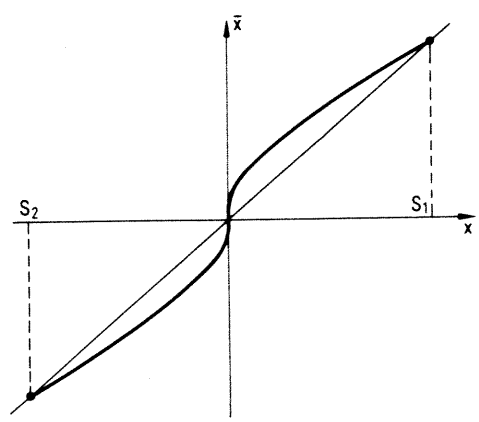

Figure 6.

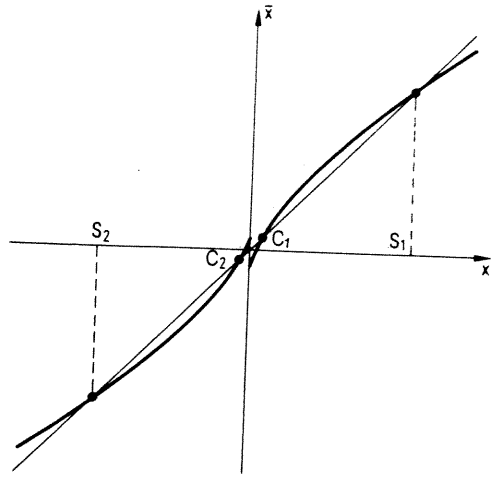

Figure 7.

From the conditions

$$\left.\frac{d\bar{x}}{dx}\right|_{x_*} = 1 = (1-\varepsilon)Ax_*^{-\varepsilon}, \qquad x = -\mu + Ax_*^{1-\varepsilon}$$

we find the equation l_2: $u = A^{1/\varepsilon} e^{-1}\varepsilon$.

Let us consider the evolution of the SA for an increase in μ. It is known that it is a structually unstable set and for $d\bar{x}/dx > 2$, simply-connected. In accordance with [1], the appearance of the lacuna and, consequently, the loss of

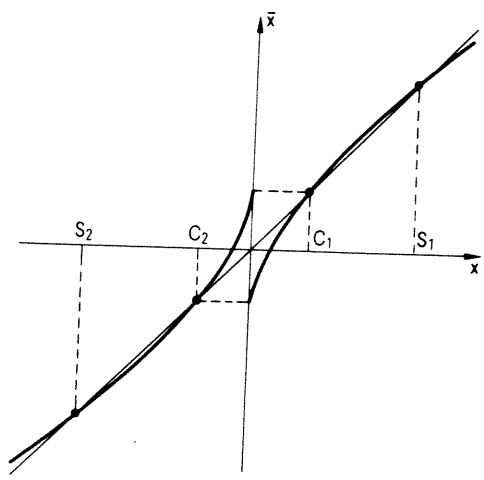

Figure 8.

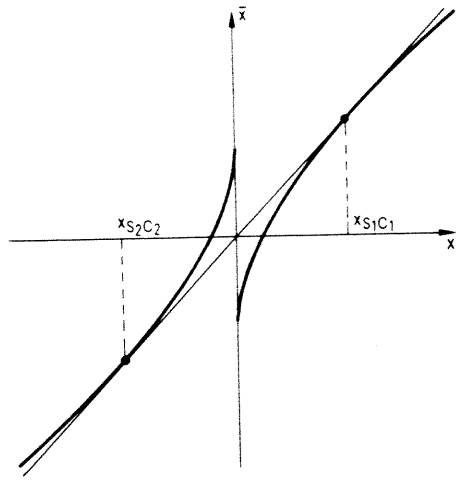

Figure 9.

simple-connectivity occurs, when the separatrices lie on a periodic motion, and the periodic motion does not have homoclinic trajectories. In the language of cascades, this means that the images of the point of discontinuity must lie on an unstable cycle G of period two with coordinates $\pm x_G$ ($x_G > 0$). Therefore, the conditions $x_G = -\mu + A_\mu^{1-\varepsilon}$ and $x_G = -(-\mu + Ax_G^{1-\varepsilon})$ must be satisfied. The bifurcation curve l_3 of the formation of the lacuna looks like this:

$$\mu = (A/\sqrt{2})^{1/\varepsilon} e^{-1}(1 + O(\varepsilon) + \cdots). \tag{5}$$

It is possible to evaluate the moment of origin of the lacuna, using the condition of violation of strong transitivity

$$\left.\frac{d\bar{x}}{dx}\right|_{x_G} = \sqrt{2} = (1-\varepsilon)Ax_G^{-\varepsilon}, \qquad x_G = -(-\mu + Ax_G^{1-\varepsilon}).$$

In this case, the equation of the bifurcation curve is similar to (5). It is easy to see that the condition $A < \sqrt{2}$ on the magnitude (we have $A < 1$) follows automatically from (5).

The next bifurcation curve is l_4; during a passage across this, the cycle G is broken into a stable cycle M and two unstable asymmetric cycles k_1 and k_2, also of period two (see Figure 10). The cycle G bifurcates when $d\bar{x}|dx|_{x_*} = 1$. Let us take advantage of the symmetry and find l_4 from the conditions

$$\left.\frac{d\bar{x}}{dx}\right|_{x_*} = -1 = -(1-\varepsilon)Ax_*^{-\varepsilon}, \qquad x_* = -(-\mu + Ax_*^{+\varepsilon}).$$

From this, we find the equation l_4: $\mu = 2A^{1/\varepsilon} e^{-1}$.

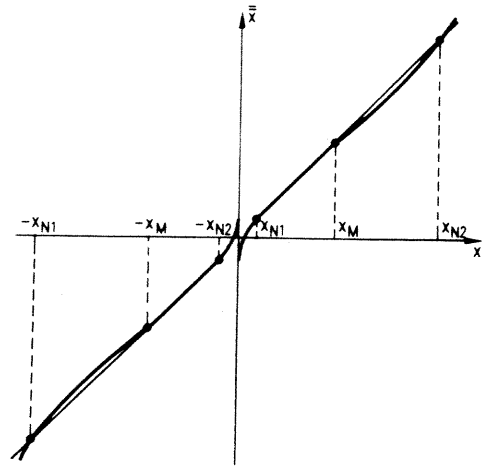

Figure 10.

The curve l_5 is the boundary of the domain of the SA. If one denotes the positive coordinates k_1 and k_2 by x_{k_1} and x_{k_2}, where $x_{k_1}(\mu) < x_{k_2}(\mu)$, then the equation of l_5 is found from the condition

$$x_{k_1}(\mu) = -\mu + A\mu^{1-\varepsilon}, \qquad x_{k_1} = -\mu + Ax_{k_1}^{1-\varepsilon}.$$

Without presenting an explicit form of l_5, let us note that it is situated between l_4 and l_6.

On l_6 the condition $T_u^2(O_\pm) = 0$ is satisfied (see Figure 11). In the language of

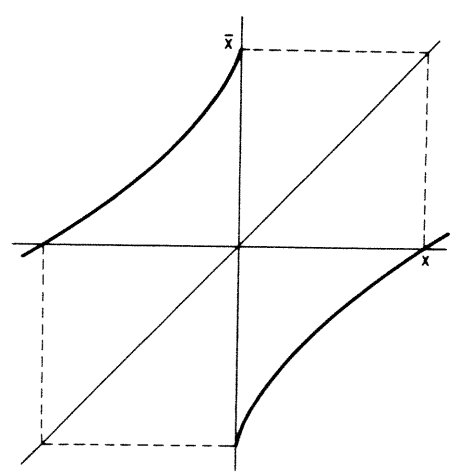

Figure 11.

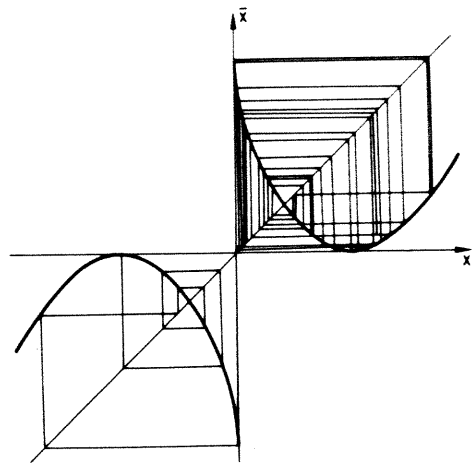

Figure 12.

flows, this shows the existence of two-circuit homoclinic curves O, into which the cycles k_1 and k_2 go. In this case, the equation of the bifurcation curve looks like this:

$$\mu = A^{1/\varepsilon}.$$

In conclusion, let us note that in the construction of bifurcation curves of periodic motions, a suite of programs [5] was used. Figures 3–11 were obtained by a numerical mapping of the transversal W^sO of a segment in a neighborhood of $\sigma = 0$ on l_1; and Figures 12 and 13, near the point $A = 0$ on l_6.

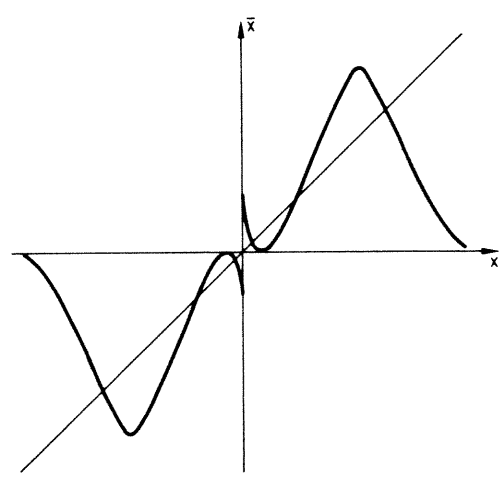

Figure 13.

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