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On bifurcations of the Lorenz attractor in the Shimizu–Morioka model

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Bifurcations of Lorenz-like attractors are studied in the Shimizu–Morioka model. Two scenarios for their destruction are suggested. The first is connected with the appearance of a lacuna, the second with a transition to a quasi-attractor. The effects of codimension two bifurcations are considered.

1. Introduction

The Shimizu–Morioka model

$$\dot{x} = y\,, \quad \dot{y} = x - \lambda y - xz\,, \quad \dot{z} = -\alpha z + x^2\,,$$
$$\alpha, \lambda > 0\,, \tag{1}$$

is considered in which complex behavior of trajectories has been discovered [1] by means of computer simulation. These equations were put forward in [1] as a model for studying the dynamics of the Lorenz system for large Rayleigh number. Some physical applications of the model were pointed out in [2]. It was shown in papers [3,4] that there are two types of Lorenz-like attractors in this model. The first is an orientable Lorenz-like attractor and the second is non-orientable containing a countable set of saddle periodic orbits with negative multipliers.

Of special interest in this model is the fact that the boundary of the region of existence of a Lorenz-like attractor includes two codimension two points (see fig. 1). The first is Q_σ ($\alpha \simeq 0.608$, $\lambda \simeq 1.044$) at which the saddle value $\sigma = -h_1 + h_3$ equals zero on the bifurcation curve l_1 corresponding to the formation of the homoclinic "figure-eight–butterfly" (here $-h_1 < -h_1 < 0 < h_3$ are eigenvalues of the origin).

The second point Q_A ($\alpha \simeq 0.549$, $\lambda \simeq 0.605$) corresponds to the vanishing of the so-called separatrix value A (for details see [5]) on the bifurcation curve l_2 corresponding to the formation of symmetric pair of double-circuit homoclinic loops Γ_1 and Γ_2. The curve l_2 is divided by Q_A into two components l_2^+ and l_2^- which are selected by conditions $A > 0$ and $A < 0$ respectively. When crossing l_2^+ (l_2^-) the saddle periodic orbits born from loops Γ_1 and Γ_2 have their invariant manifolds homeomorphic to a cylinder (to a Möbius strip), respectively.

It was stated by Šil'nikov [6] that under some additional restrictions on the eigenvalues of a multi-dimensional saddle, there exist regions of parameter values in neighbourhoods of such points for which systems close to the symmetric system with two homoclinic loops have a Lorenz-like attractor[#1]. An appropriate Poincaré map was shown in [6] to have a one-dimensional

[#1] In the case $n \geq 4$ the appearance of a Lorenz-like attractor may also occur when the separatrices return along the non-leading manifolds to the saddle tangential to each other [6].

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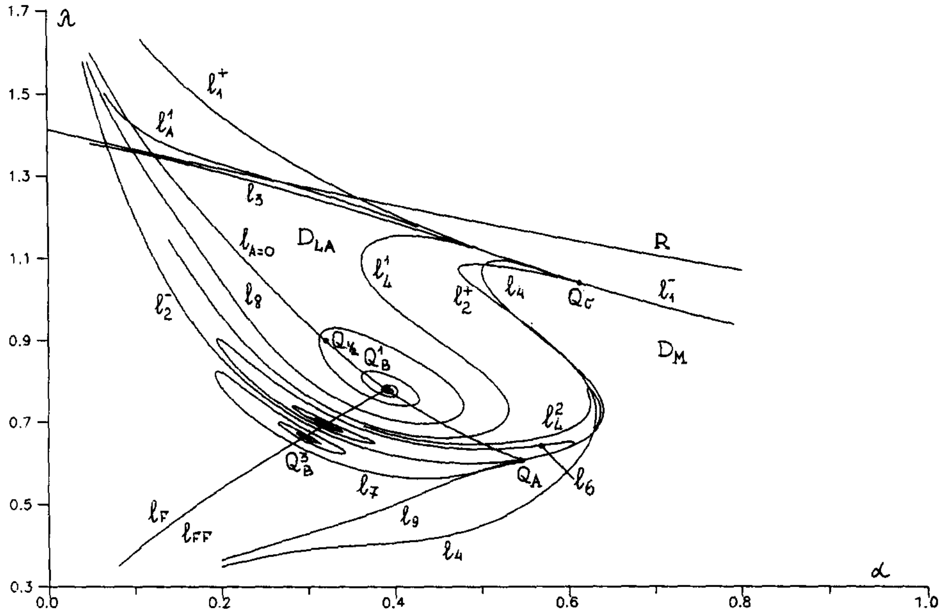

Fig. 1. A (α, λ) bifurcation diagram of the Shimizu–Morioka system. Bifurcation curves are explained in the text.

discontinuous map as a "normal form". In some small neighbourhood (in general depending on σ) of the point Q_σ this map is of the form[#2]

$$x = (-\mu_1 + A|x|^{1-\mu_2})\operatorname{sgn}(x)\,, \tag{2}$$

where $\mu_1 \ll 1$, $\mu_2 = \sigma/h_3 \ll 1$; and in neighborhood of the point Q_A there is the form

$$x = (-\mu_1 + \mu_2|x|^\gamma + |x|^\nu)\operatorname{sgn}(x)\,, \tag{3}$$

where $\mu_2 = A \ll 1$, $0 < \gamma = h_1/h_3 < 1$ is the saddle index, and $\nu = \min\{2h_1/h_3, h_2/h_3\}$. In papers [3,8], an analysis of bifurcations near such points was given and the main bifurcation curves were extended into the (α, λ)-parameter plane for eqs. (1) by means of qualitative-numerical methods. It should be noted that map (2) was studied there only for $0 < A < 1$ because the value of A obtained numerically is approximately 0.81 at Q_σ. Similar maps were also considered by Lyubimov and Zaks [9], and Glendinning [10].

[#2] This reduction can be made based on Robinson's results [7] on the existence of a smooth C^1-filiation in the region of the Lorenz-like attractor.

In the present paper the study of the Shimizu–Morioka model is continued with a more detailed analysis of bifurcations near one of the branches of the boundary of the region of existence of the Lorenz-like attractor, where a transition to quasi-attractors is observed.

2. Bifurcations between points Q_σ and Q_A

The system (1) is invariant with respect to the substitution $(x, y, z) \to (-x, -y, z)$. For $\alpha, \lambda > 0$, it possesses three equilibrium states: $O = (0,0,0)$ is a saddle of the type (2.1) i.e. $\dim W_0^s = 2$, $\dim W_0^u = 1$, $W_0^u = \Gamma_1 \cup O \cup \Gamma_2$ where Γ_1 and Γ_2 are one-dimensional separatrices coming out from O as $t \to \infty$; $O_{1,2} = (\pm\alpha^{1/2}, 0, 1)$ are both either stable foci or saddle-foci of type $(1,2)$[#3]. The boundary of stability of $O_{1,2}$ is the

[#3] We introduce a coding of separatrix Γ_1. It is a sequence (π) of units and zeros: "0" is written if Γ_1 goes around the one-dimensional stable invariant manifold of equilibrium state O_1, "1" – around O_2. A coding $(\bar{\pi})$ may be obtained from (π) by the replacement: $1 \Leftrightarrow 0$. When Γ_i forms a homoclinic loop, the coding is finite and we shall use the notation Γ_i^n. Analogously codings for periodic orbits are introduced.

curve R: $\alpha = (2-\lambda^2)/\lambda$. The characteristic equation for $O_{1,2}$ on R has a pair of pure imaginary roots. It is well known that the character of the bifurcation in such a case depends on the sign of the first Lyapunov value Z_1. Here Z_1 is negative for $\alpha \geq 0.089$.

The point Q_σ divides l_1 into the two components l_1^- and l_1^+ satisfying conditions $\sigma < 0$ and $\sigma > 0$, respectively. Thus there are two cases in the study of dynamics of the system (1) near l_1.

The case $\sigma < 0$. For α, λ lying above R equilibrium states O_1 and O_2, are stable foci and attract the separatrices Γ_1 and Γ_2. When crossing R stable periodic orbits S_1 and S_2 are born from O_1 and O_2, respectively, and Γ_1 and Γ_2 tend to them (fig. 2a). The size of S_1 and S_2 grows as parameters α and λ approach l_1^- and the become the stable homoclinic figure-eight–butterfly on l_1^- itself (fig. 2b). After passing through l_1^- a stable symmetric orbit M_1 appears (fig. 2c). It is the unique stable limit set in domain D_M of fig. 1.

The case $\sigma > 0$. The point Q_σ is a limit point for the curves in fig. 1 which correspond to:

l_A^1 and l_A^2 – the boundaries of the region of existence of the Lorenz-like attractor (the curve l_A^2 is indistinguishable from curve l_2 in fig. 1 and is not separately labelled);

l_L^1 – the appearance of a simple lacuna[#4];

l_2 – the formation of the double-circuit homoclinic loops Γ_1^{01} and Γ_2^{10};

l_3 – a saddle–node bifurcation of periodic orbits involving S_1 and S_2;

l_4 – a pitch–fork bifurcation of periodic orbits involving M_1.

[#4] The spectral decomposition theorem [11] establishes that the non-wandering set of Lorenz-like flows consists of the Lorenz attractor and a set $\Sigma = \cup_{i=0}^{N} \Sigma_i$, where Σ_i is a hyperbolic periodic orbit or a non-trivial hyperbolic set conjugate to a subshift of finite type. When the well-known condition of complete dilation [5] is satisfied Σ is empty. Otherwise, this is not the case. Each Σ_i is situated in a so-called lacuna – a "hole" in the attractor. In our case, a saddle symmetric figure-eight periodic orbit lies in a lacuna (see fig. 3c).

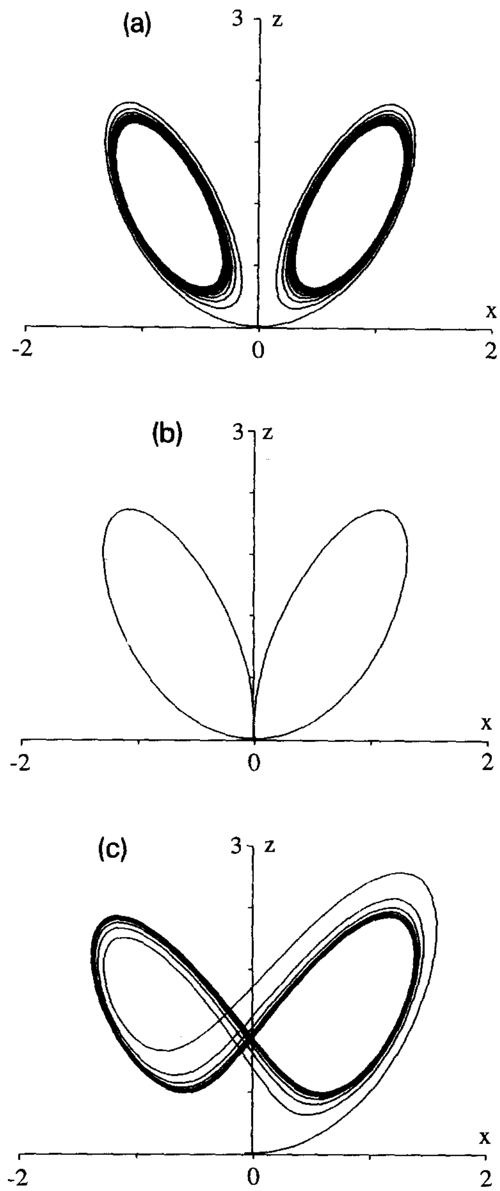

Fig. 2. (a) Separatrices Γ_1 and Γ_2 are attracted by stable periodic orbits S_1 and S_2, respectively ($\alpha \simeq 0.8$, $\lambda \simeq 1.0$); (b) a homoclinic figure-eight–butterfly ($\alpha \simeq 0.8$, $\lambda \simeq 0.934$); (c) a stable symmetric periodic orbit M_1 ($\alpha \simeq 0.8$, $\lambda \simeq 0.8$).

Let us describe the evolution of limit sets in the phase space of eqs. (1) when moving transversely to these curves.

Along l_1^+ all the necessary conditions for the Šil'nikov theorem are satisfied: the separatrices

Γ_1 and Γ_2 return to the origin 0 tangent to the leading direction (the z-axis) and the separatrix value A is positive. Therefore, after crossing l_1^+ towards domain D_{LA}, saddle periodic orbits C_1 and C_2 are born from the homoclinic figure-eight–butterfly together with a limit set Ω_{B} such that system (1) on it is homeomorphic to a suspension over the Bernoulli shift on two symbols [5]. Above l_{A}^1 this set is non-attracting, and so Γ_1 and Γ_2 tend either to the stable foci O_2 and O_1 (above the curve R) or to the stable periodic orbits S_2 and S_1 (below the curve R) (fig. 3a). The set Ω_{B} becomes the Lorenz-like attractor after crossing l_{A}^1 on which $\Gamma_1 \in W_{C_2}^s$ and $\Gamma_2 \in W_{C_1}^s$ (fig. 3b), where $W_{C_{1,2}}^s$ are the two-dimensional stable invariant manifolds of $C_{1,2}$. There exist three attractors in the region between l_{A}^i and l_3. The first one is the Lorenz-like attractor, the two others are either $O_{1,2}$ or $S_{1,2}$ above and below the curve R, respectively. On l_3 the stable periodic orbits $S_{1,2}$ coalesce in pairs with the saddle orbits $C_{1,2}$ and disappear. In the domain D_{LA} the Lorenz-like attractor is the unique stable set and consists of one connected component.

Let us now consider an evolution of the Lorenz-like attractor when moving from domain D_{LA} to D_M through $l_{\mathrm{L}}^1, l_4, l_{\mathrm{A}}^2, l_2$. It was noted above that the separatrix value A is positive and less than unity. Therefore, a bifurcation in which a lacuna appears will precede the destruction of the Lorenz-like attractor. The lacuna arises when $\Gamma_{1,2}$ lie on the stable manifold of a symmetric saddle periodic orbit G_1. Fig. 3c shows the (x–z)-projection of the Lorenz-like attractor with the simple lacuna containing no non-wandering orbits except for G_1. When crossing l_4 the orbit

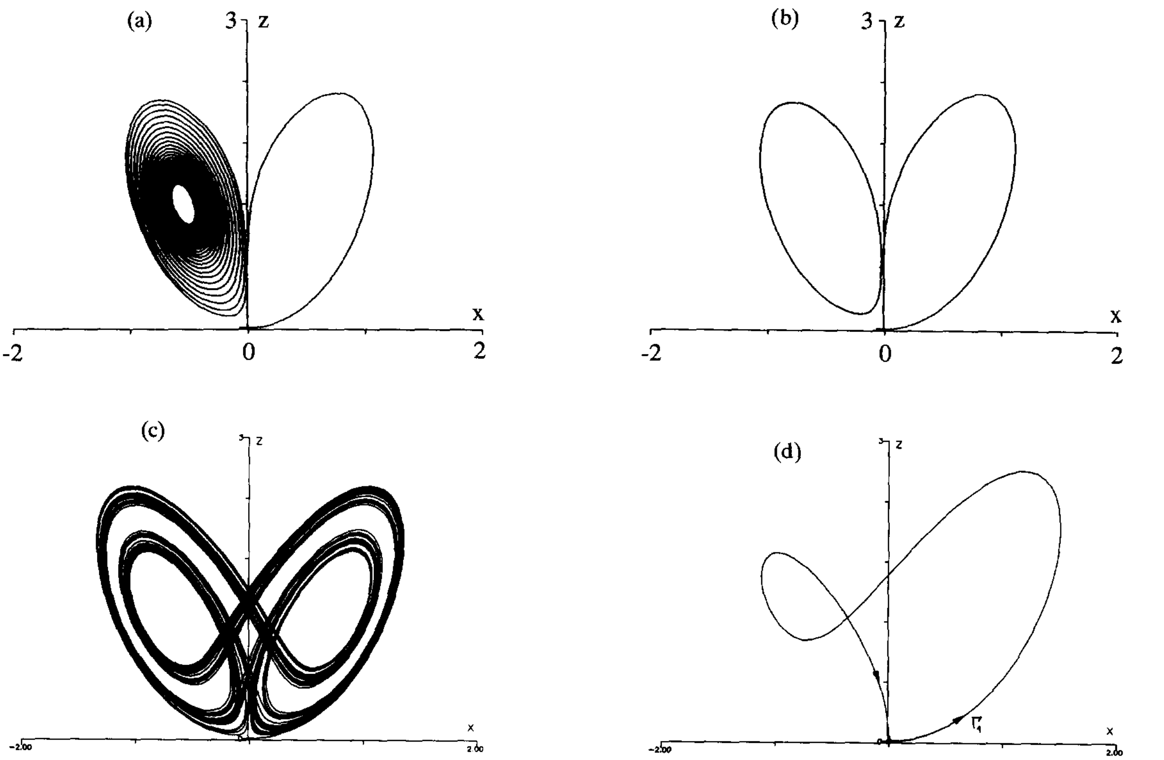

Fig. 3. (a) The separatrix Γ_1 tends to a stable periodic orbit S_2 ($\alpha \simeq 0.35$, $\lambda = 1.24$); (b) the moment of the appearance of a Lorenz-like attractor: Γ_1 lies on the stable invariant manifold of the saddle periodic orbit C_2 ($\alpha \simeq 0.2$, $\lambda \simeq 1.33$); (c) a Lorenz-like attractor with a simple lacuna containing no non-wandering set except a saddle symmetric periodic orbit G_1 ($\alpha \simeq 0.5$, $\lambda \simeq 0.85$); (d) a double-circuit homoclinic loop Γ_1^{01} ($\alpha \simeq 0.928$, $\lambda \simeq 0.555$).

G_1 inside the lacuna bifurcates into three: the stable symmetric periodic orbit M_1 and two saddle non-symmetric periodic orbits K_1^{01} and K_2^{10}. The last determine the boundary of the basin of M_1. The Lorenz-like attractor disappears after crossing l_A^2 on which there are the inclusions $\Gamma_1 \in W^s(K_2^{01})$ and $\Gamma_2 \in W^s(K_1^{10})$. The difference between bifurcations on l_A^2 and l_A^1 is only that on l_A^2 the non-symmetric double-circuit periodic orbits K_1^{01} and K_2^{10} play the role of the simple periodic orbits C_1 and C_2. To the right of l_A^2 the separatrices Γ_1 and Γ_2 are attracted to the stable periodic orbit M_1. On l_2, K_1^{01} and K_2^{10} become the homoclinic loops Γ_1^{01} and Γ_2^{10} (fig. 3d).

The bifurcation sequence becomes slightly more complicated if we go from D_M to D_{LA} through l_2, l_A^2, l_4, l_5. On l_4 the stable symmetric periodic orbit M_1 loses its stability, being divided into the saddle periodic orbit G_1 and two stable non-symmetric ones N_1^{01} and N_2^{10} having the

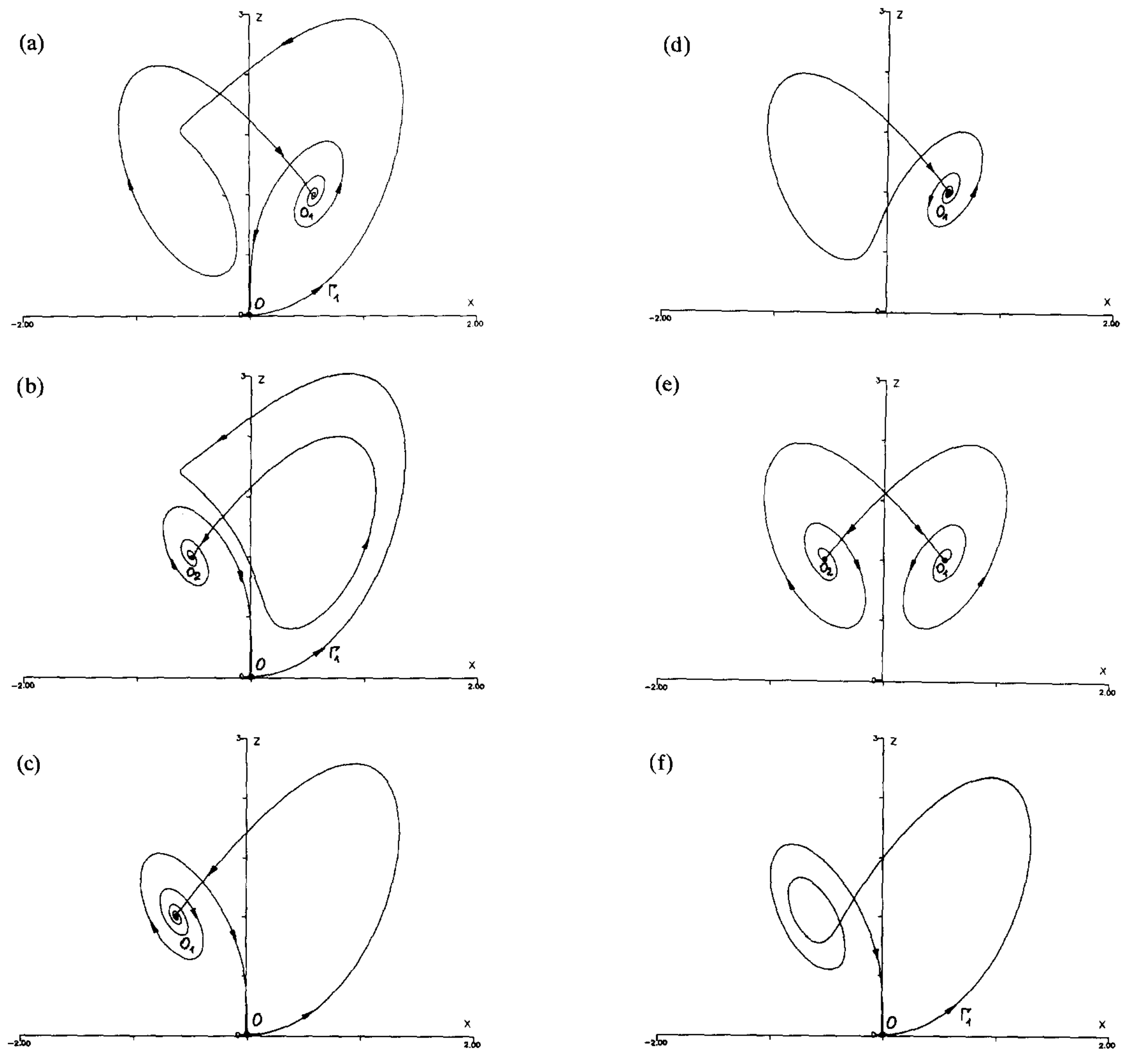

Fig. 4. Heteroclinic contours at the codimension two points (a) Q_B^2 ($\alpha \simeq 0.316$, $\lambda \simeq 0.699$); (b) Q_B^3 ($\alpha \simeq 0.3$, $\lambda \simeq 0.660$); (c) Q_B^1 ($\alpha \simeq 0.390$, $\lambda \simeq 0.781$); (d) a homoclinic loop to the saddle-focus O_1 ($a^* \simeq 0.23368867$, $\lambda \simeq 0.7$); (e) a heteroclinic contour between saddle-foci O_1 and O_2 ($\alpha \simeq \alpha^* + 0.23 \times 10^{-5}$, $\lambda \simeq 0.7$); (f) a homoclinic loop Γ_1^{011} ($\alpha \simeq 0.476$, $\lambda \simeq 0.723$.

same shapes as K_1^{01} and K_2^{10}, respectively. After crossing l_A^2 there exist already five periodic orbits G_1, $N_{1,2}$, $K_{1,2}$ in the lacuna. On l_5 non-symmetric orbits coalesce in pairs into two non-rough orbits of saddle–node type and disappear. The curve l_5 starts from the codimension two point Q_ζ at which the Lyapunov value ζ_2 equals zero on l_4. (Neither Q_ζ nor l_s are included in fig. 1).

Notice the following bifurcation curves originating from Q_A:

l_A^2 and l_A^3 are the boundaries of the region of existence of the Lorenz-like attractor. On l_A^3, Γ_1 (Γ_2) belongs to a stable invariant manifold of the saddle non-symmetric periodic orbits with the coding: (1001) ((0110));

on l_L^2, $\Gamma_{1,2}$ tend to the saddle symmetric periodic orbits G_2 with the coding (0110). It is a moment when a second lacuna appears.

l_6 and l_7 correspond to the formation of homoclinic loops with codings (0110) and (0101), respectively;

l_9 corresponds to two non-rough non-symmetric periodic orbits $N_{1,2}$ with multipliers equal to -1.

The curves l_6 and l_7 are remarkable in that they are spirals winding to the points Q_B^2 and Q_B^3, respectively. Both these points belong to a codimension two set and correspond to the formation of heteroclinic contours that have been studied in [12,13] (fig. 4a,b). It follows from [12] that there exists a countable set of points similar to $Q_B^{2,3}$ accumulating at the point Q_B^1 ($\alpha \simeq 0.3903$, $\lambda \simeq 7807$) (fig. 4c). All points of this set lie in a domain bordered by the curves l_F and l_{FF} (in fig. 1 curves l_F and l_{FF} are indistinguishable because $\mathrm{dist}(l_F, l_{FF}) \sim 10^{-4}$). The curve l_F corresponds to the formation of homoclinic loops to saddle-foci $O_{1,2}$ (fig. 4d) and l_{FF} to a heteroclinic contour including both O_1 and O_2 (fig. 4e). These curves terminate at the point Q_B^1. The curve l_8 winds into Q_B^1, and on it there are homoclinic loops with codings (011) and (100) (fig. 4f). According to [12], on l_8 there exists a countable set $\{Q_A^i\}$ of points at each of which the separatrix value A vanishes. These points are the ones of an intersection of l_8 with the curve $l_{A=0}$, which goes through Q_B^1 and finishes at Q_A. On $l_{A=0}$ a change of sign of the separatrix value A occurs, that is geometrically accompanied by the formation of distinctive "bends" in the Poincaré map (see fig. 5).

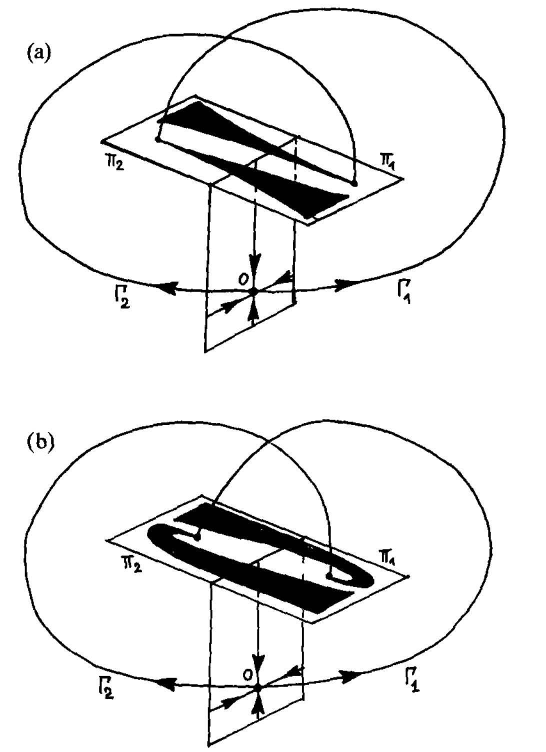

Fig. 5. First return map for the plane $\Pi = \Pi_1 \cup \Pi_2$. Trajectories started within Π_1 and Π_2 next strike the plane within the shaded area for parameters α, λ (a) above and (b) below the curve $l_{A=0}$ in fig. 1.

3. Boundary between Lorenz attractor and quasi-attractor

Obviously, behavior of orbits of eqs. (1) on $l_{A=0}$ depends on the sign of the value $\nu - 1$ (see (3)). In our case $\nu = 2\gamma$. Thus $l_{A=0}$ is divided by the point $Q_{1/2}$ into two components defined by conditions $\gamma < 1/2$ and $\gamma > 1/2$. The bifurcations occurring when passing through the left branch are the same ones as in the Lorenz model [14].

Namely, structurally unstable homoclinic orbits arise below $l_{A=0}$ that cause stable periodic orbits to appear in the attractor of system (1). Such complex limit sets are quasi-attractors in the terminology of [11,15].

For $\gamma < 1/2$, the region of existence of the Lorenz-like attractor adjoins the curve $l_{A=0}$, but this is not the case for $\gamma > 1/2$. Here dynamics of the system (1) is well modelled by the map

$$x = (-\mu_1 + |x|^{2\gamma})\,\mathrm{sgn}(x)\,, \tag{4}$$

but not only for small x. Such a map was considered by Arneodo, Coullet and Tresser [16] and Lyubimov and Zaks [9]. It was stated in these papers that if the homoclinic loops with coding (π) (and therefore symmetric to them loops with a coding $(\bar{\pi})$) exist for some value μ_0, then a sequence $\mu_0 < \mu_1 < \mu_2 < \cdots < \mu_\infty$ exists such that:

(i) for $\mu = \mu_i$ a pair of homoclinic (π_i)-and $(\bar{\pi}_i)$-loops exists where

$$\pi_0 = \pi\,, \quad \pi_{i+1} = \pi_i \bar{\pi}_i\,; \tag{5}$$

(ii) for $\mu_i < \mu < \mu'_i$ a stable symmetric periodic (π_{i+1})-orbit exists which loses its stability at $\mu = \mu'_i < \mu_{i+1}$. A pair of non-symmetric stable (π_{i+1})-orbits bifurcate off from it at μ'_i;

(iii) at $\mu = \mu_{i+1}$ these orbits become homoclinic $(\pi_i \bar{\pi}_i)$- and $(\bar{\pi}_i \pi_i)$-loops, respectively. On further increasing μ a stable symmetric $(\pi_i \bar{\pi}_i \bar{\pi}_i \pi_i)$-orbit is born from this homoclinic figure eight.

Thus the selected bifurcations of "period-doubling" of the loops are analogous to a Feigenbaum cascade. At $\mu = \mu_0$ the closure of the separatrices is a Feigenbaum-like set. Under this construction each μ_i corresponds to a homoclinic figure-eight–butterfly with vanishing separatrix value A. Analysis of map (3) shows that a bundle of bifurcation curves originates from each such point. We select the following curves to mention here (see fig. 6): l_{π_i} – a formation of loops with codings π_i and $\bar{\pi}_i$;

$l^{\pi_i}_{\mathrm{LA}}$ – the first boundary of the region of existence of the Lorenz-like attractor. The coding of the separatrix Γ_1 is $\pi_i(\bar{\pi}_i)^\infty$ on it;

$l^{\pi_i}_{\mathrm{sn}}$ – a saddle–node bifurcation of non-symmetric periodic orbits;

$l^{\pi_i}_{\mathrm{L}}$ – an appearance of a lacuna on a symmetric periodic orbit with coding $\pi_i \bar{\pi}_i$;

$l^{\pi_i}_{\mathrm{pf}}$ – a subcritical pitch–fork bufurcation of the periodic $\pi_i \bar{\pi}_i$-orbit;

$l^{\pi_{i+1}}_{\mathrm{LA}}$ – the second boundary of the region of existence of the Lorenz-like attractor (the separatrix Γ_1 has a coding $\pi_i(\bar{\pi}_i \pi_i)^\infty$ here);

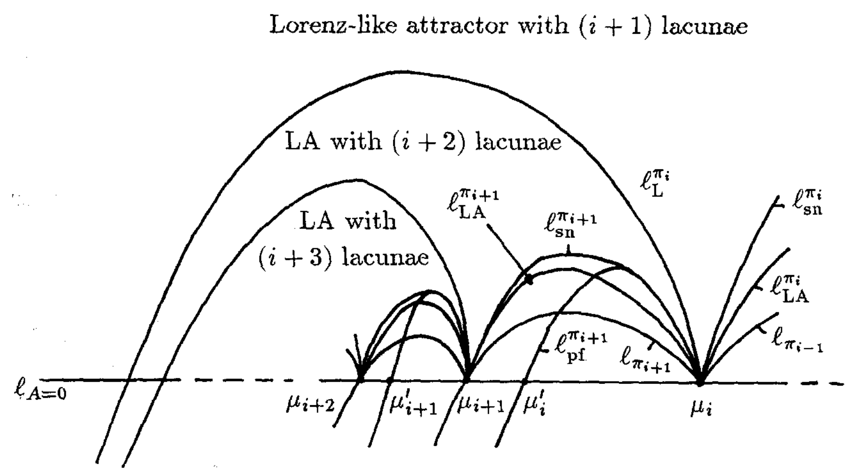

Fig. 6. The structure of the boundary of the region of existence of a Lorenz-like attractor near the curve $l_{A=0}$.

$l_{\pi_{i+1}}$ – an existence of double-circuit homoclinic $\pi_i\bar{\pi}_i$- and $\bar{\pi}_i\pi_i$-loops.

It is necessary to fit these bundles together – "to play a bifurcation patience" by Šil'nikov's expression. The simplest variant is presented in fig. 6. Here, there are points $Q_Z^{\pi_i}$ where the Lyapunov values equal zero. To the left of each such point there is a supercritical pitch-fork bifurcation. The curves $l_L^{\pi_i}$ and $l_{\pi_{i+1}}$, $\pi_i = (01)$, $\pi_{i+1} = (0110)$, in fig. 6 are l_L^2 and l_6 in fig. 1, respectively. Thus, the boundary of the region of existence of a Lorenz-like attractor is the

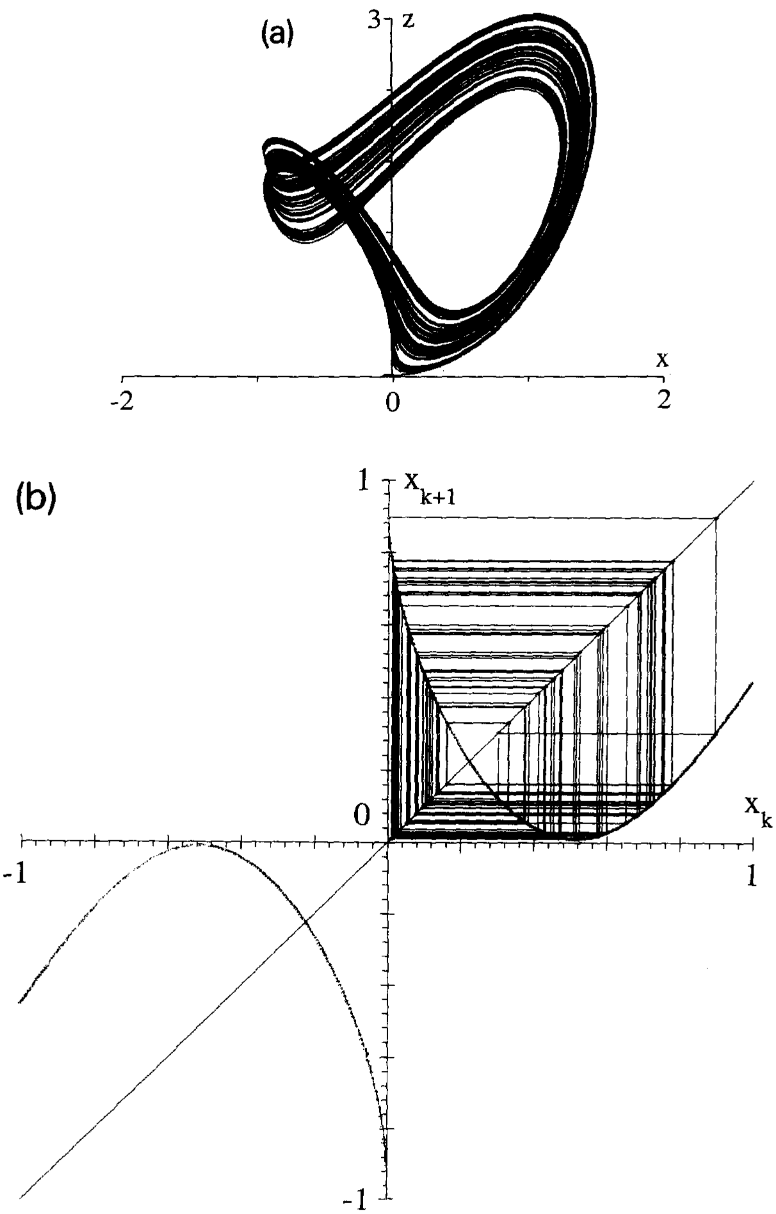

Fig. 7. (a) A non-symmetric quasi-attractor for eqs. (1), and (b) corresponding to the one-dimensional return map.

union of arches of $l_{LA}^{\pi_i+1}$ under all admissible codings π_i. Note that the mechanism of destroying the Lorenz-like attractor realized in the domain between μ_i and μ_{i+1} is analogous to the one that is considered above for the domain between Q_σ and Q_A.

As stated in [8], narrow sectors of existence of the non-orientable Lorenz attractors adjoin to each point μ_i from the domain where $A<0$. However, attractors are more typically quasi-attractors in this domain. The period-doubling bifurcation for periodic orbits N_1^{01} and N_2^{10} is the first bifurcation in a cascade of period-doublings leading to the formation of two non-symmetric quasi-attractors. Fig. 7a shows the $(x-z)$-projection of the "right" quasi-attractor for $\alpha=0.45$, $\lambda=0.555$. When increasing the parameter λ both quasi-attractors unite into a single unique symmetric quasi-attractor. It is easy to understand this phenomenon by considering the Poincaré map. The strong contraction causes this map to be extremely close to some one-dimensional discontinuous map. Fig. 7b represents the graph of this map obtained numerically for $\alpha=0.45$, $\lambda=0.555$. One can see that the uniting of quasi-attractors occurs when extrema of the graph lie on the x-axis.

Acknowledgements

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Note added

More details on the behaviour of the Shimizu–Morioka model can be found in the independent work of A.M. Rucklidge [17].

References

[1] T. Shimizu and N. Morioka, Phys. Lett. A76 (1980) 201.
[2] A. Rucklidge, Poster NATO Adv. Res. Workshop in Brussels (1991).
[3] A.L. Šil'nikov, Sel. Math. Sov. 10 (1991) 105 (originally published in: Methods of Qualitative Theory of Differential Equations, Gorky St. Univ. (1986), p. 180).
[4] A.L. Šil'nikov, Methods of Qualitative Theory and Theory of Bifurcations, Gorky St. Univ. (1989), p. 130.
[5] V.S. Afraimovich, V.V. Bykov and L.P. Šil'nikov, Trans. Moscow Math. Soc. 2 (1983) 153.
[6] L.P. Šil'nikov, Usp. Math. Nauk. 36 (1981) 240 (Russ. Math. Surv. in English).
[7] C. Robinson, Nonlinearity 2 (1989) 495.
[8] A.L. Šil'nikov, Ph.D. Thesis, University of Gorky (1990).
[9] D.V. Lyubimov and M.A. Zaks, Physica D9 (1983) 52.
[10] P. Glendinning, Phys. Letters A 121 (1987) 411.
[11] V.S. Afraimovich and L.P. Šil'nikov, Nonlinear Dynamics and Turbulence (Pittman, New York, 1983), p. 1.
[12] V.V. Bykov, Meth. Qual. Theory Dynam. System (1980) (in Russian) p. 44–72.
[13] P. Glendinning and C. Sparrow, J. Stat. Phys. 43 (1986) 479.
[14] V.V. Bykov and A.L. Šil'nikov, Sel. Math. Sov., to appear (originally published in :Methods of Qualitative Theory and Theory of Bifurcations, Gorky St. Univ. (1989), p. 151).
[15] L.P. Šil'nikov, Diff. Eq. 17 (1981) 2105 (in Russian).
[16] A. Arneodo, P. Coullet and C. Tresser, Phys. Lett. A 81 (1981) 197.
[17] A.M. Rucklidge, Physica D 62 (1993) 323, these Proceedings.