Blue Sky Catastrophe in Two Time Scale Systems

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Brief Introduction

- *Period* & *Length* classification of known codimension-1 bifurcations of stable periodic orbits;

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- Scenario of Blue Sky Catastrophe;
- Realization: exemplary model;

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- Candidate: Hindmarch-Rose model of neuron activity

Behavior of a stable periodic orbit on stability boundary

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- Group III : period $T \to \infty$, whereas length L is finite

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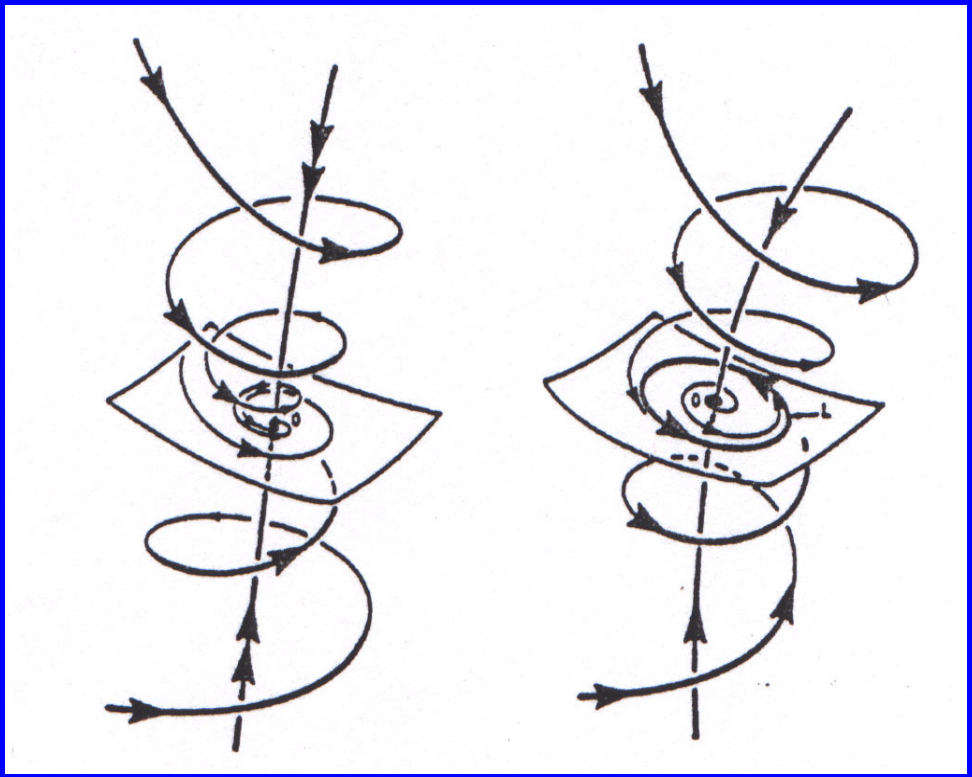

Figure 1: Supercritical Andronov-Hopf bifurcation

Group II : both T and L finite

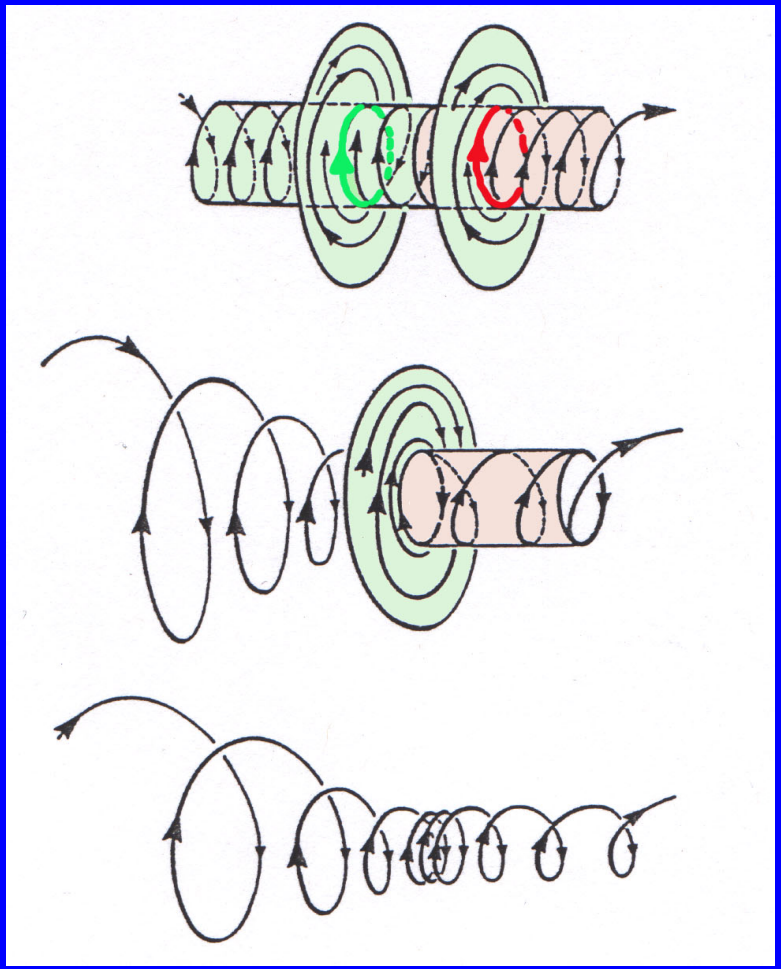

Figure 2: Saddle-node limit cycle, $m = +1$

Group II : both T and L finite

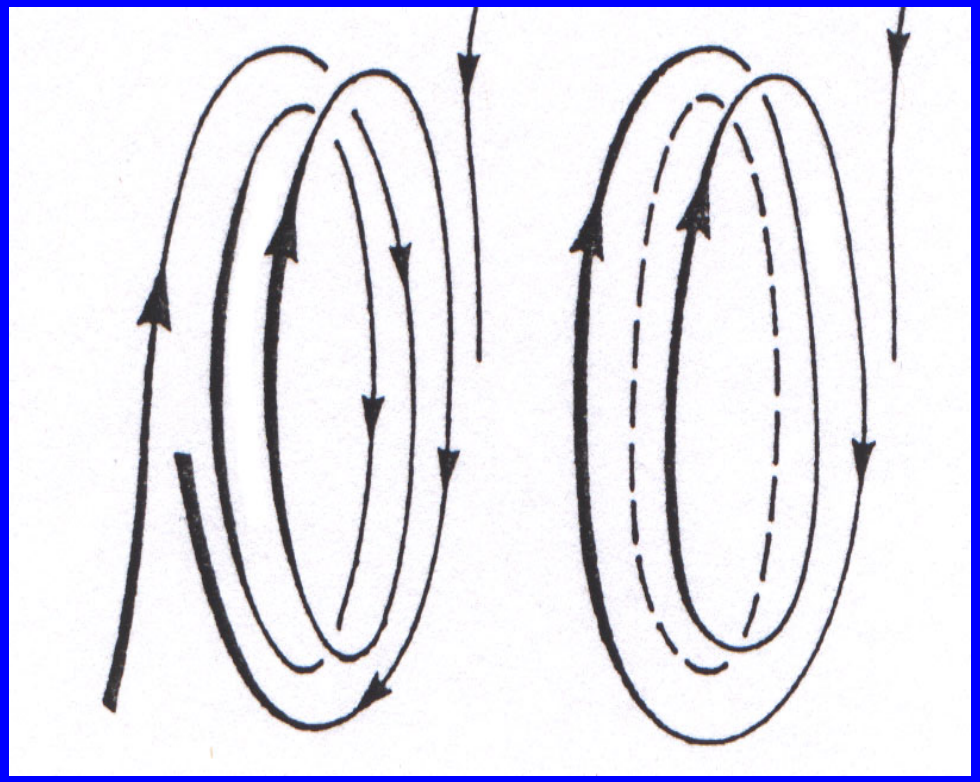

Figure 3: Saddle-node limit cycle, $m = -1$

Group II : both T and L finite

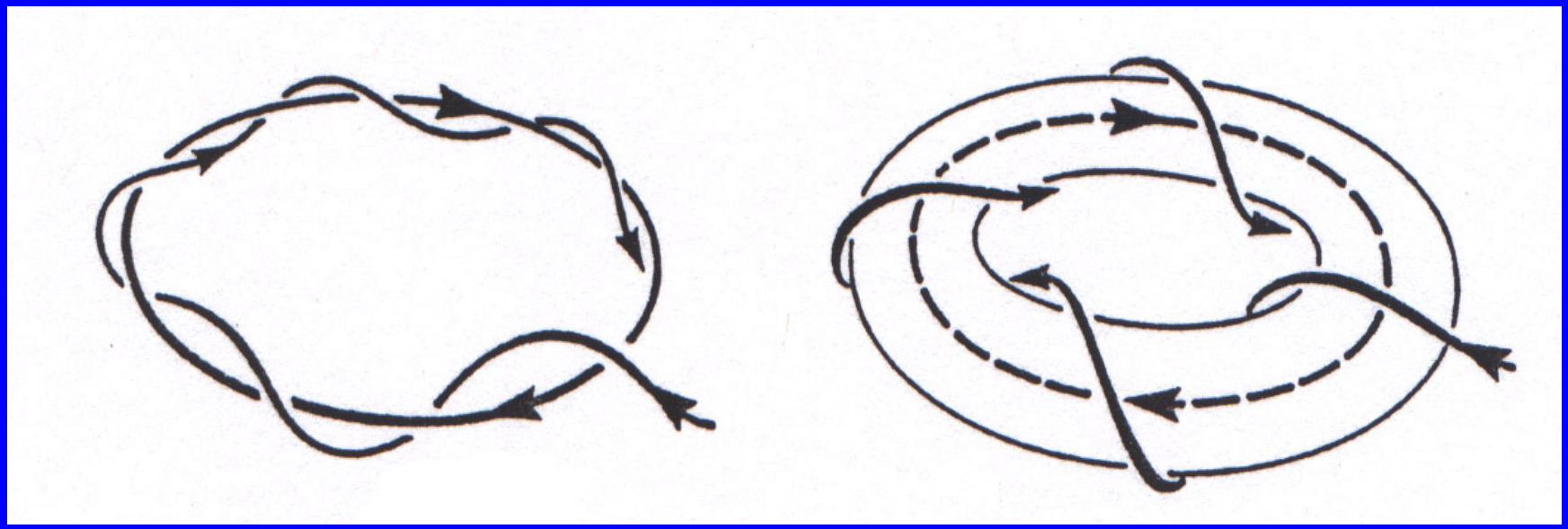

Figure 4: Torus (non-resonant) bifurcation , $m_{1,2} = e^{\pm i\omega}$

Group III : $T \to \infty$, finite L

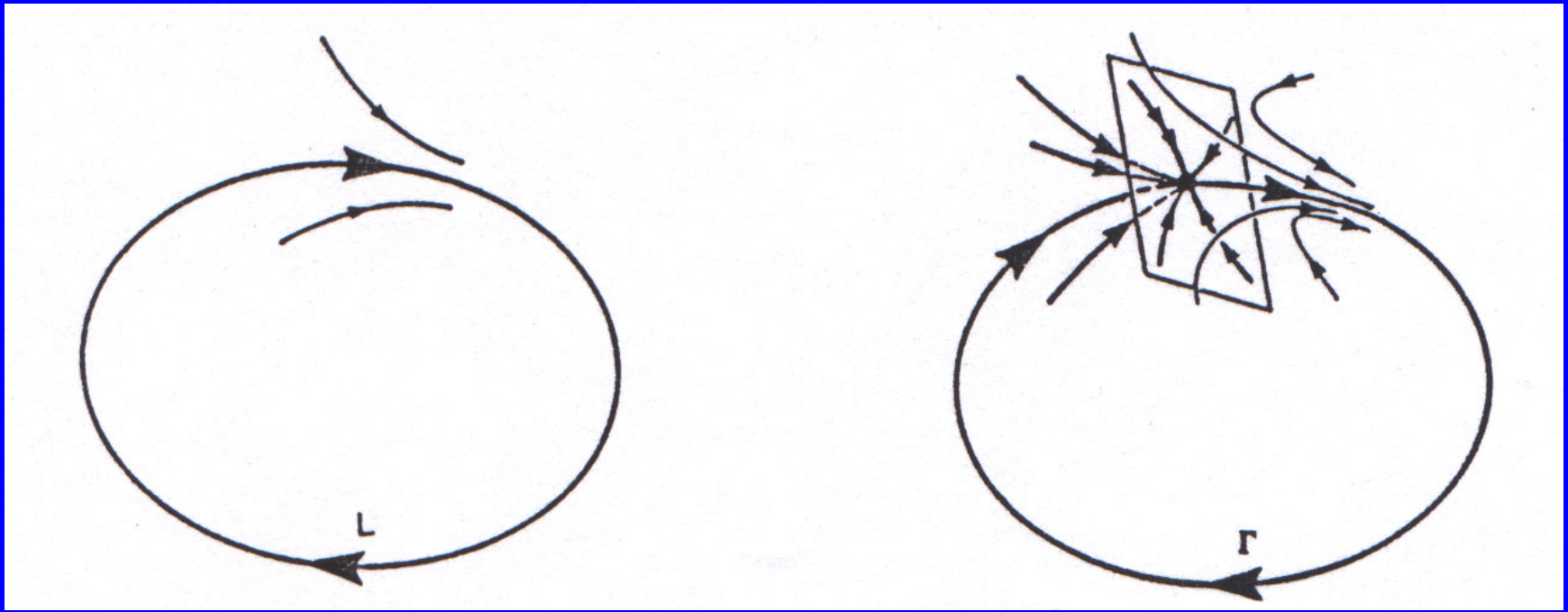

Figure 5: Homoclinic orbit to saddle-node point

Group III : $T \to \infty$, finite L

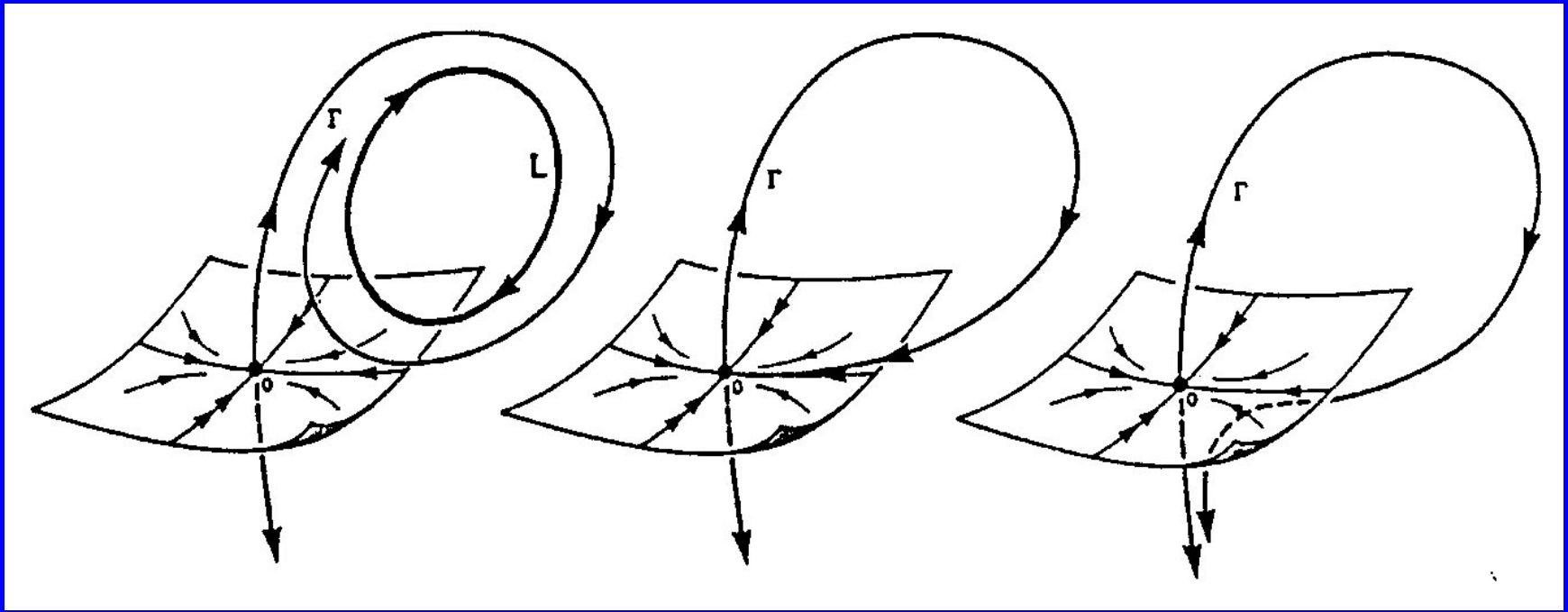

Figure 6: Homoclinic orbit to a saddle point

Palis and Pugh problem

Is there yet a new bifurcation of a stable limit cycle that meets the following conditions:

1. Codimension 1;

2. no bifurcations before it reaches the stability boundary

3. no equilibria (where $\dot{X} = 0$)

Hypothetic example I

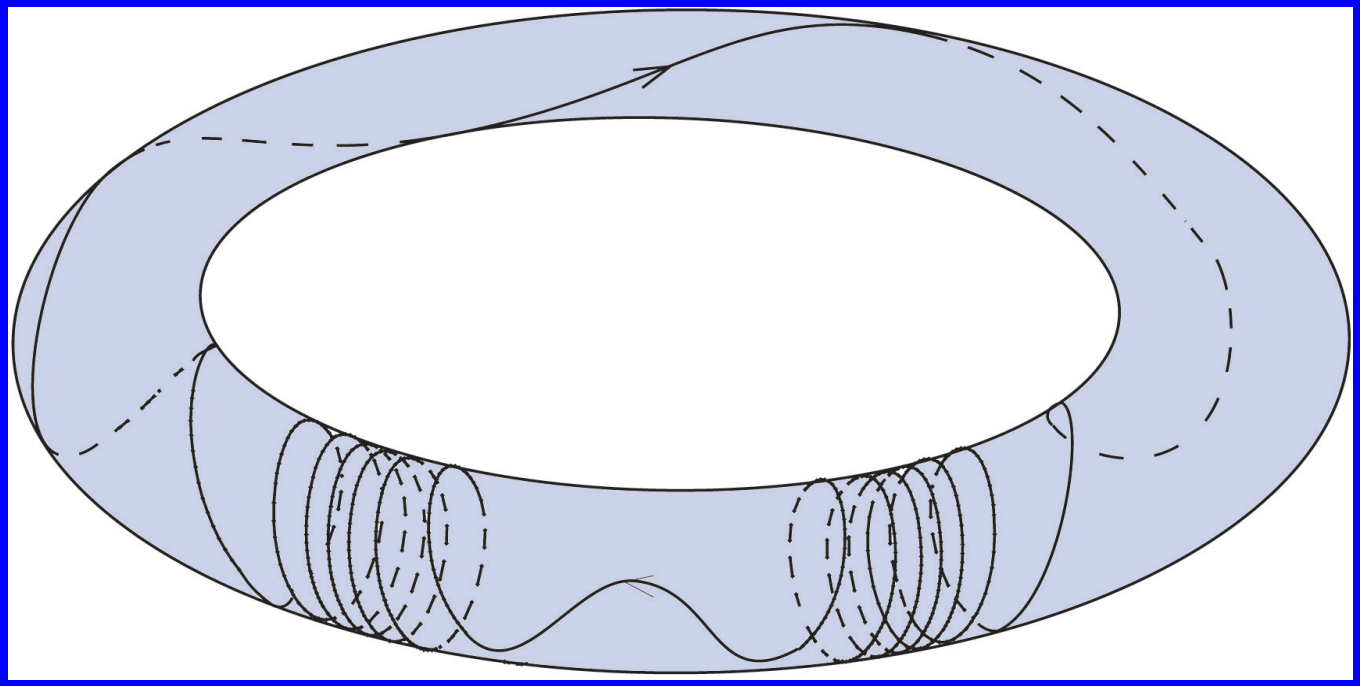

Figure 7: Special resonant torus

Shilnikov-Turaev construction

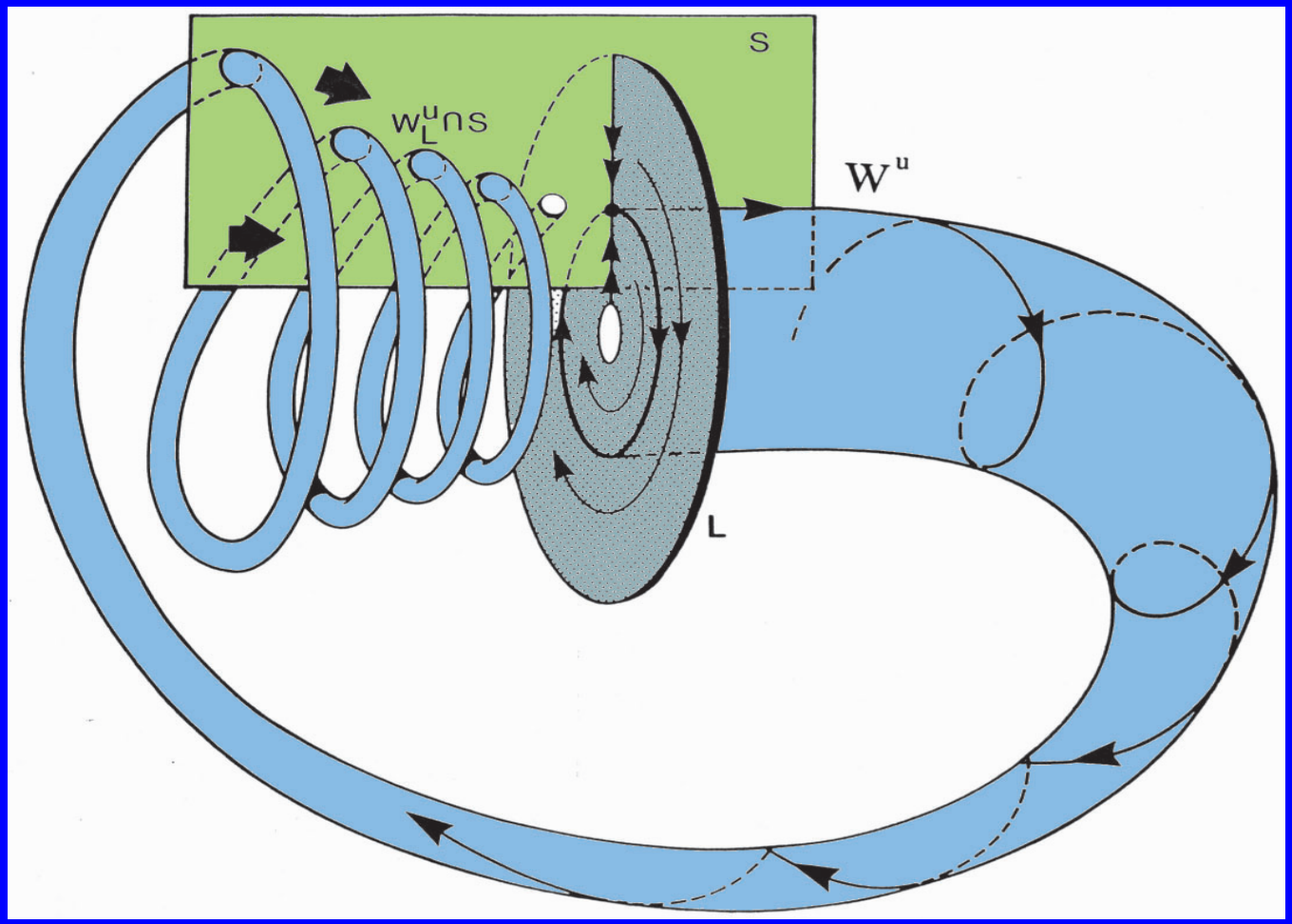

Figure 8: Blue sky bifurcation

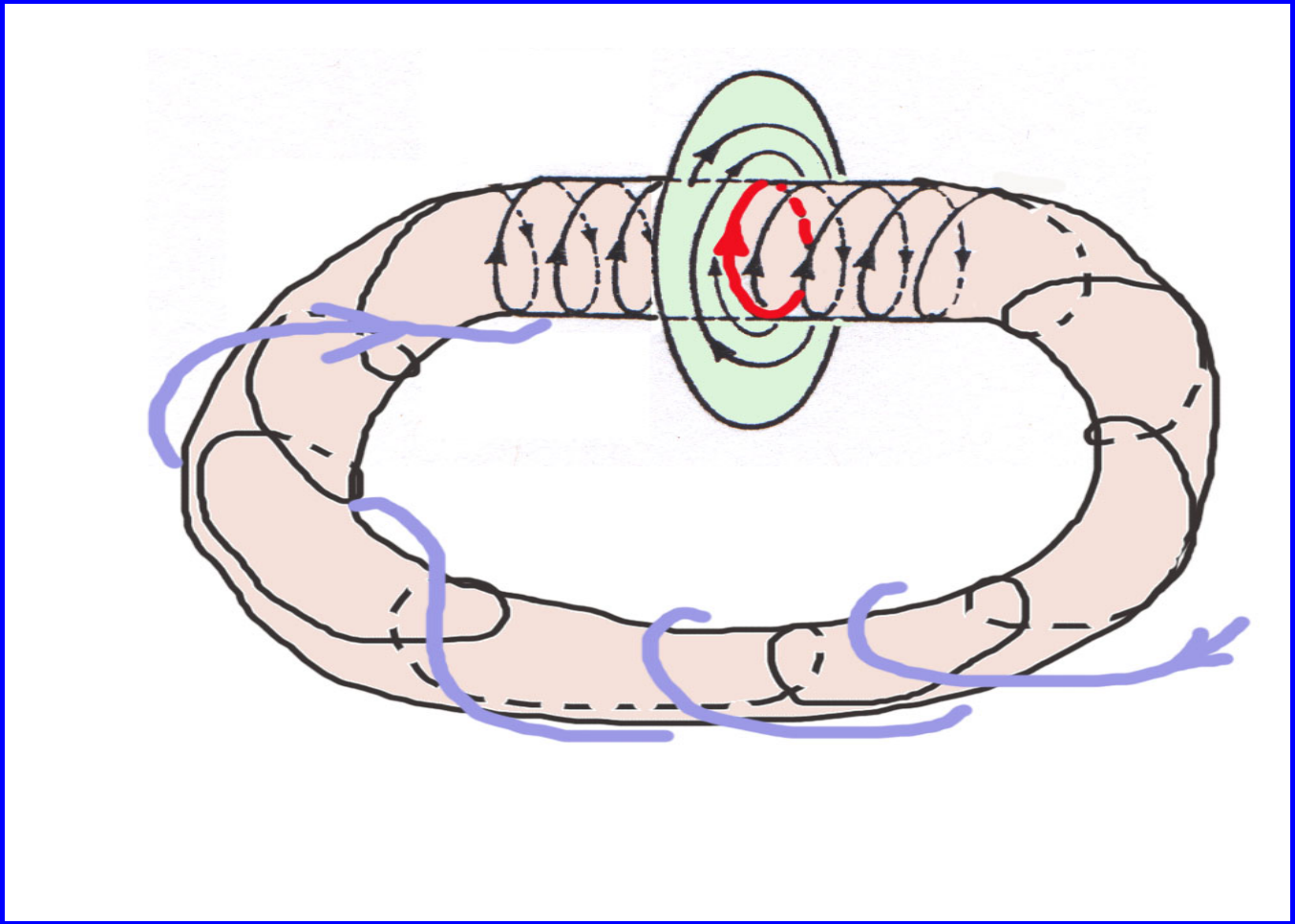

Figure 9: Homoclinic Saddle-node cycle $\Rightarrow$ Resonant torus

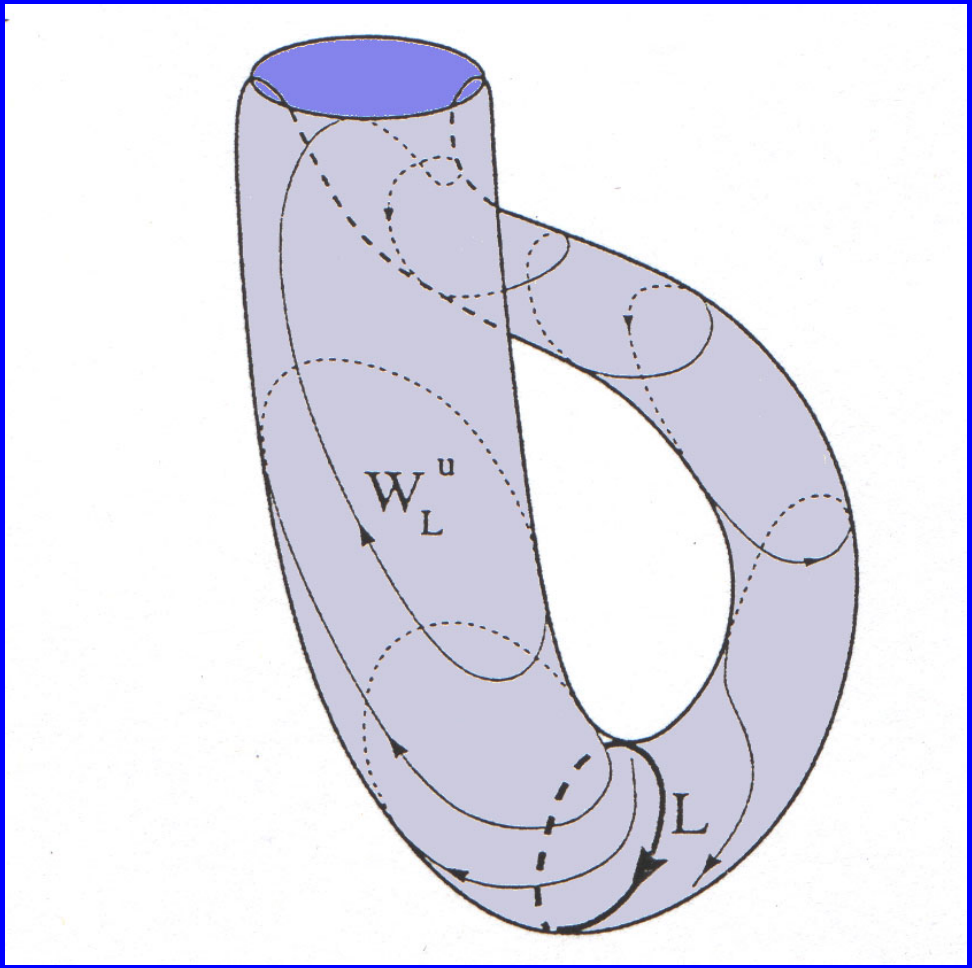

Figure 10: Klein bottle

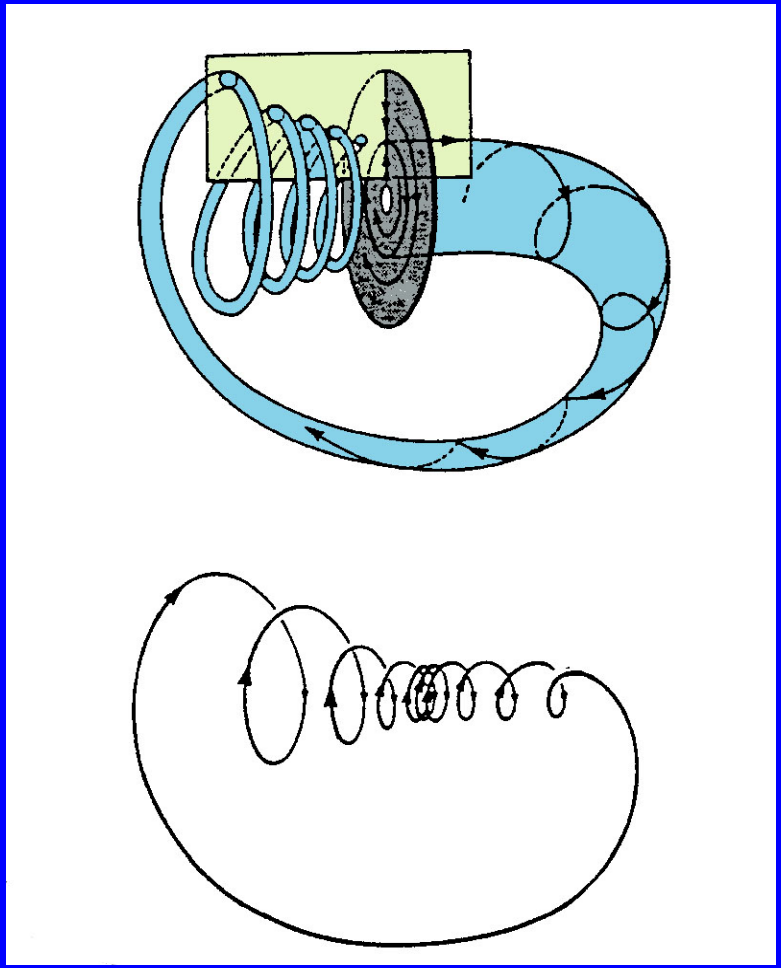

Figure 11: Blue sky catastrophe in action

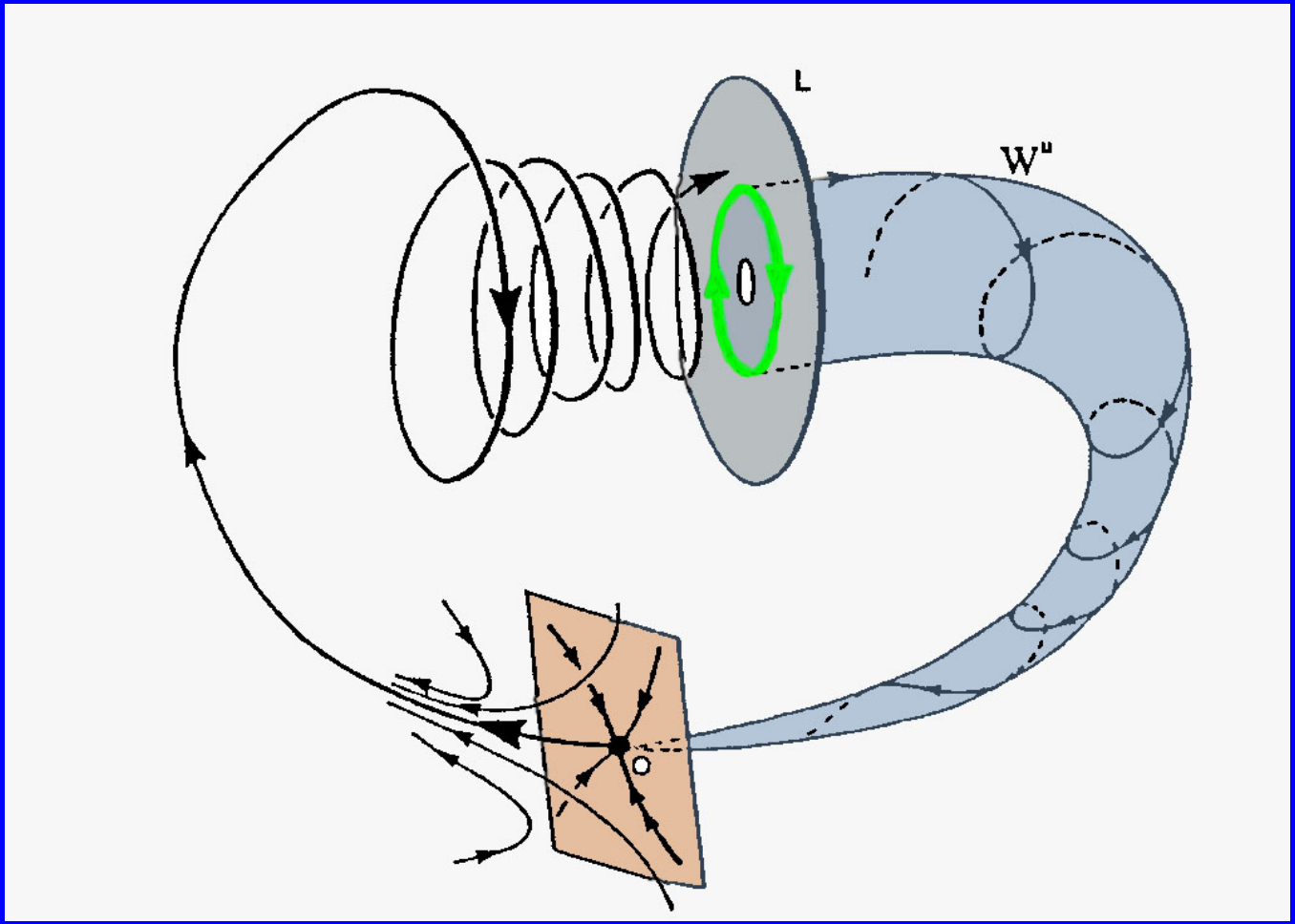

Figure 12: Contraction and squeezing

Gavrilov-Shilnikov normal form

$$\begin{cases} \dot{x} = x(2+\mu - B(x^2+y^2)) + z^2 + y^2 + 2y, \\ \dot{y} = -z^3 - (1+y)(z^2+y^2+2y) - 4x + \mu y, \\ \dot{z} = (1+y)z^2 + x^2 - \varepsilon, \end{cases} \tag{1}$$

where μ, ε, and B $(= 10)$ are parameters.

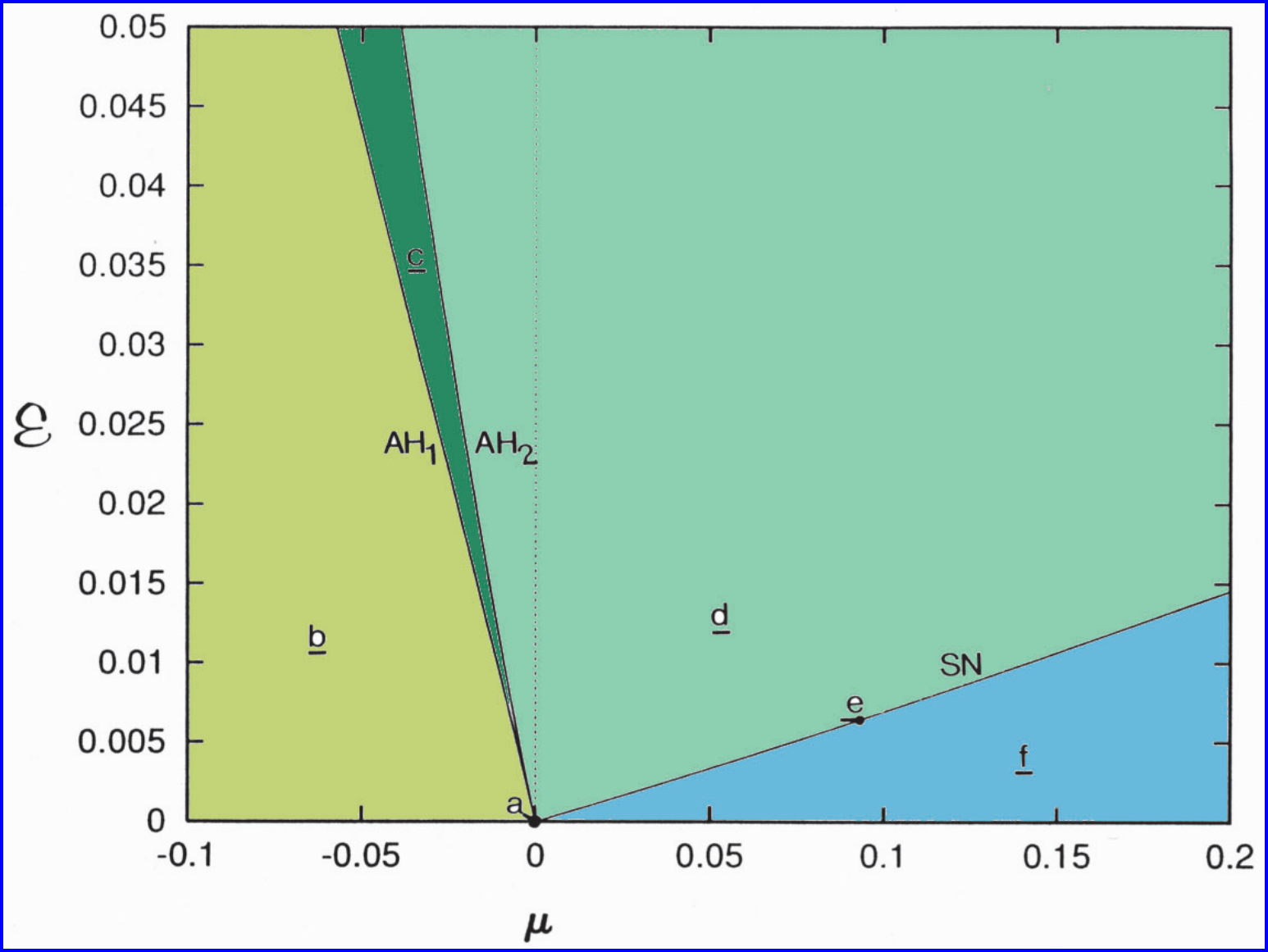

Figure 13: Bifurcation diagram

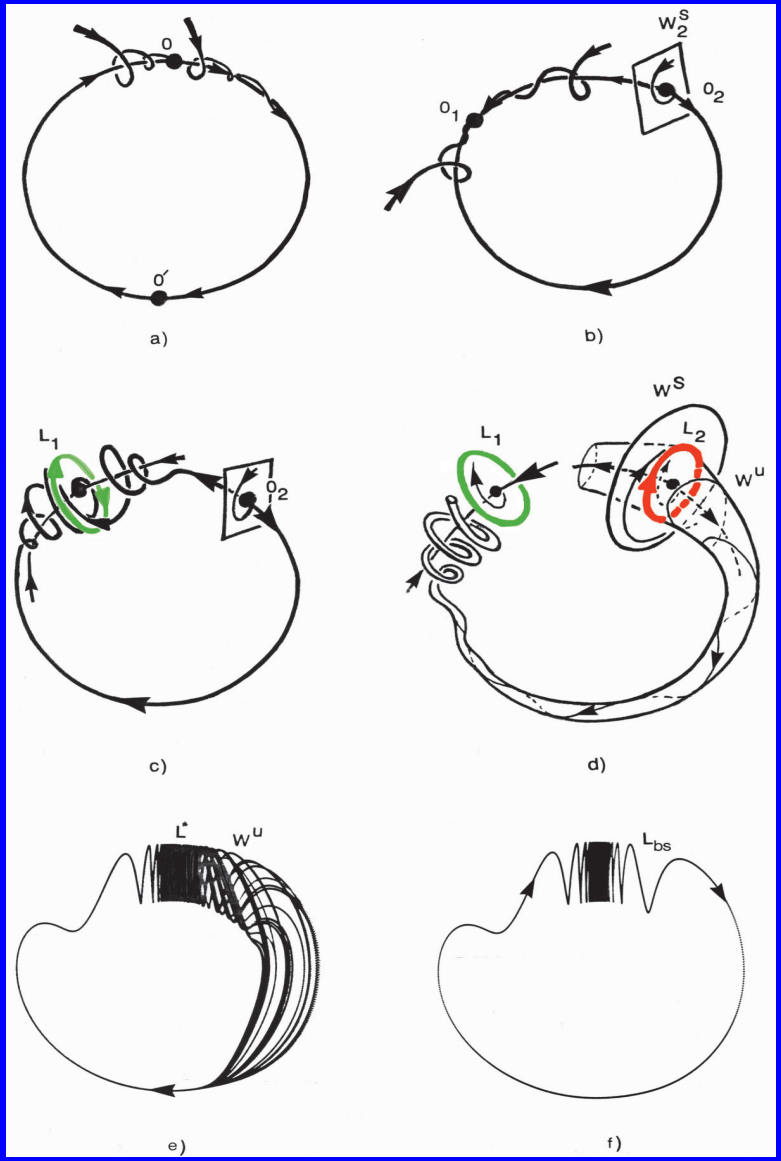

Figure 14: Phase portraits

Two time scale system

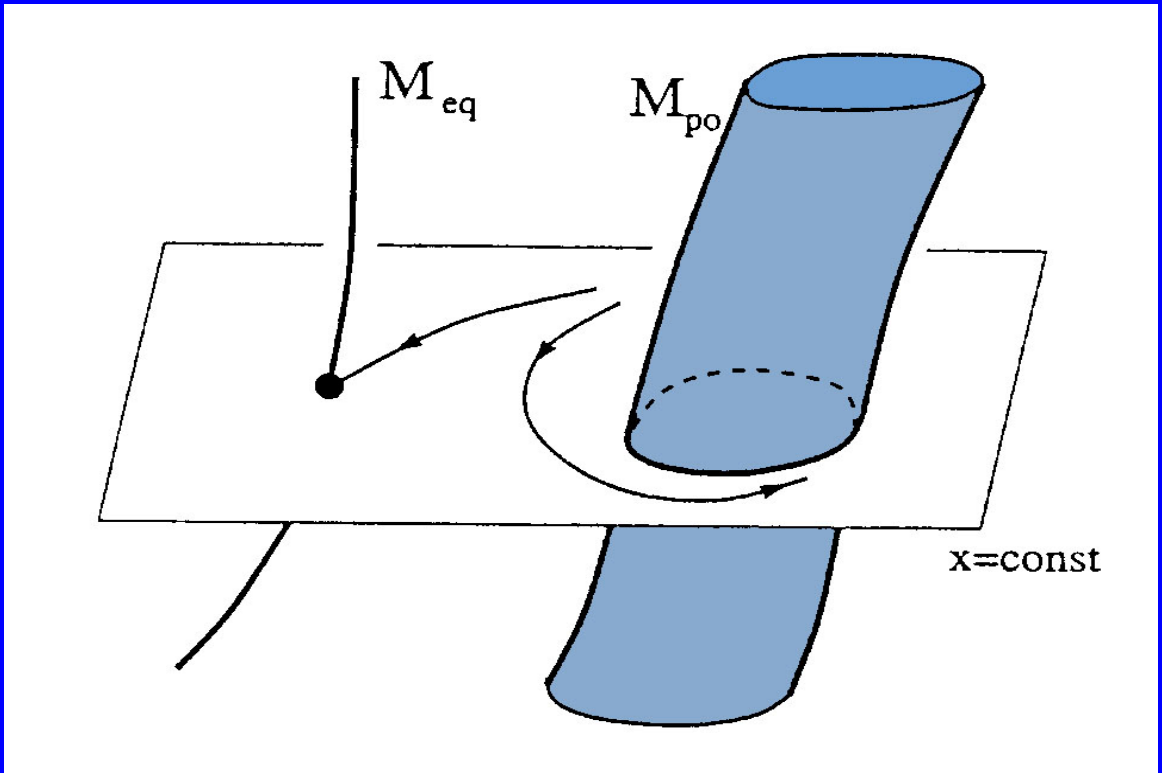

Figure 15: Primary Ingredients

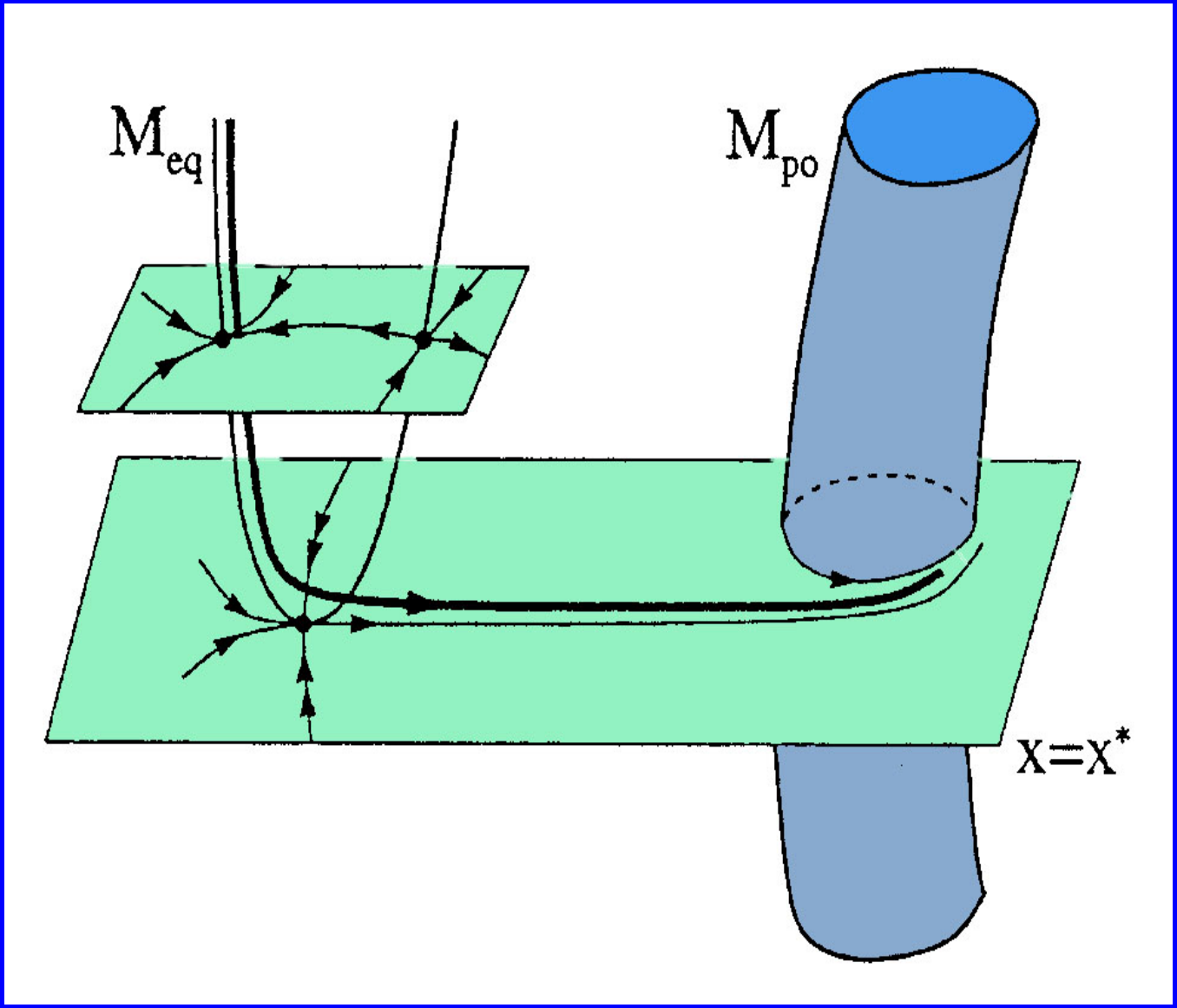

Figure 16:

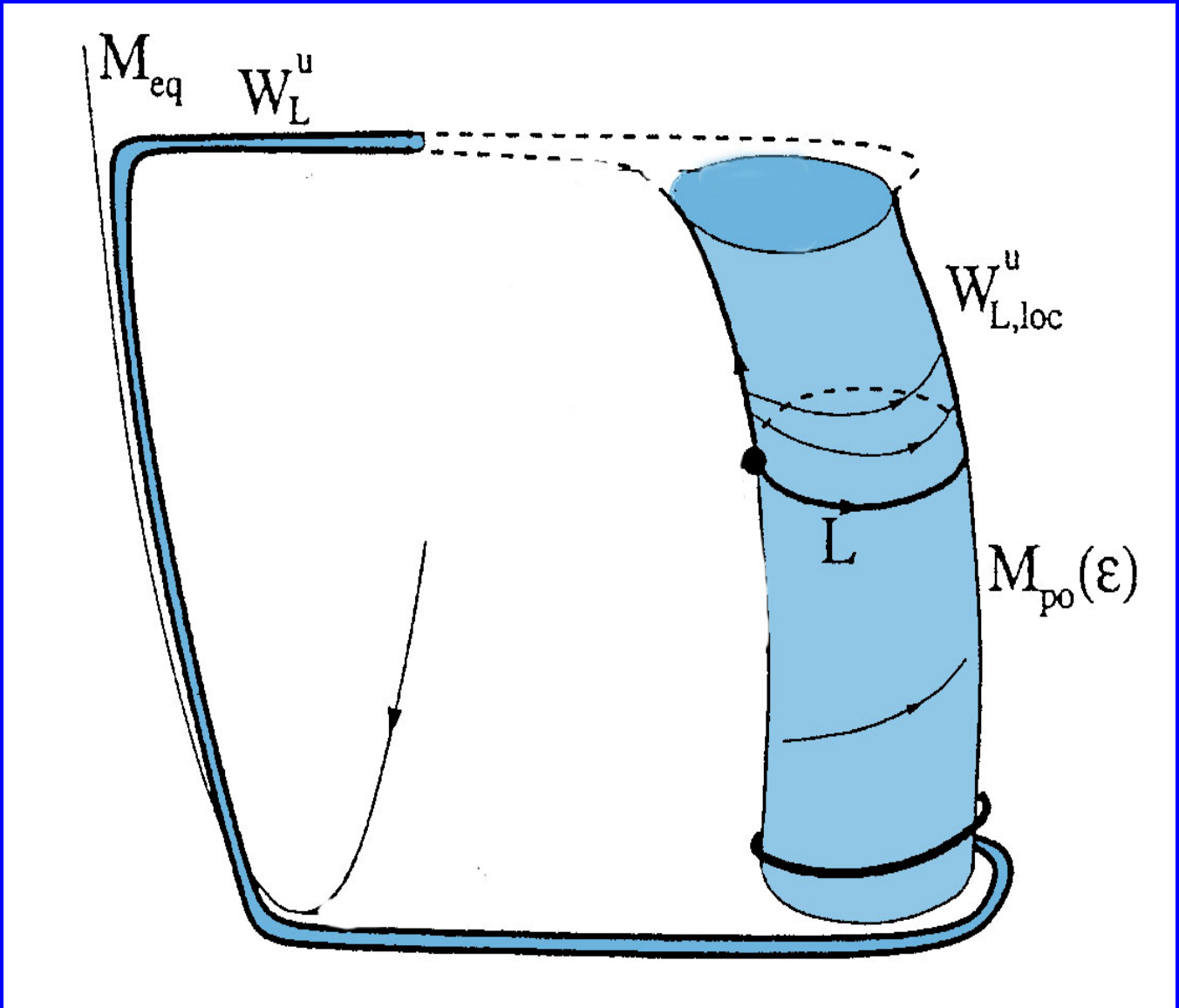

Figure 17: Primary Ingredients

Hindmarsh-Rose model of neuron activity

$$\begin{cases} \dot{x} = y - z - x^3 + 3x^2 + 5, \\ \dot{y} = -y - 2 - 5x^2, \\ \dot{z} = \varepsilon(2(x + 2.1) - z). \end{cases} \tag{2}$$

Slow sub-system

$$\begin{cases} \dot{x} = y - z - x^3 + 3x^2 + 5, \\ \dot{y} = -y - 2 - 5x^2, \end{cases} \tag{3}$$

where z is a parameter.

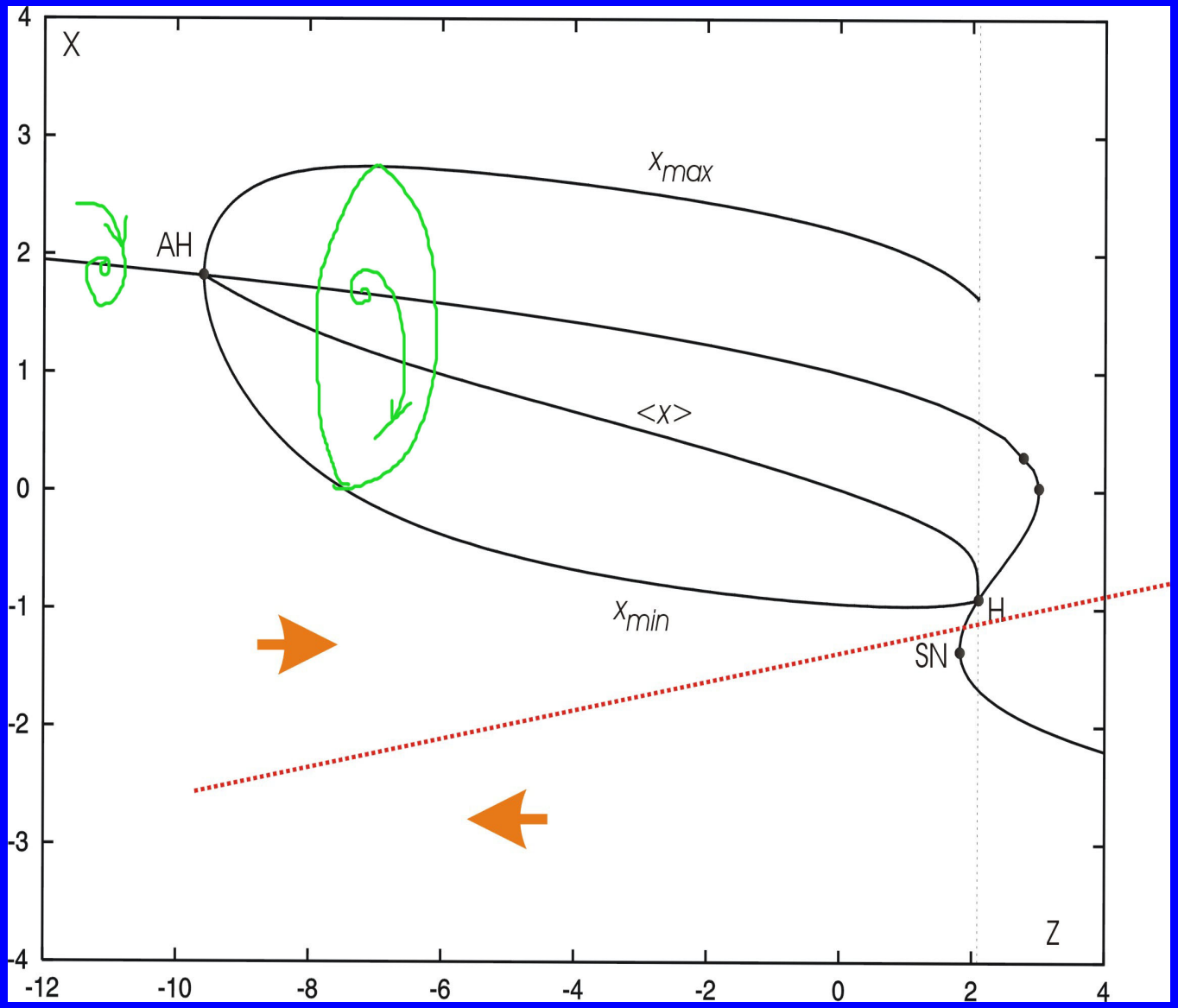

Figure 18: Bifurcation diagram for slow sub-system

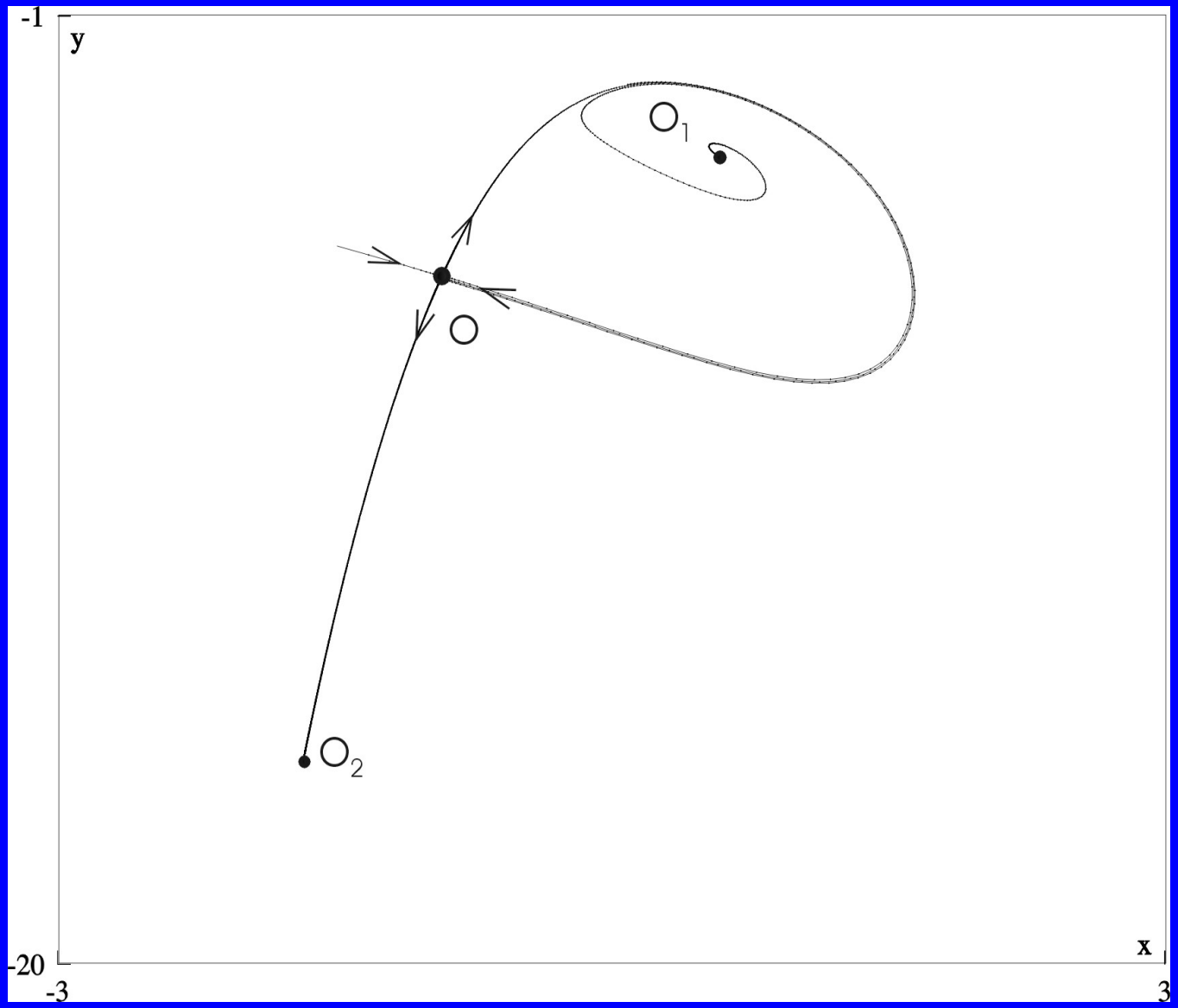

Figure 19: Homoclinics in slow system

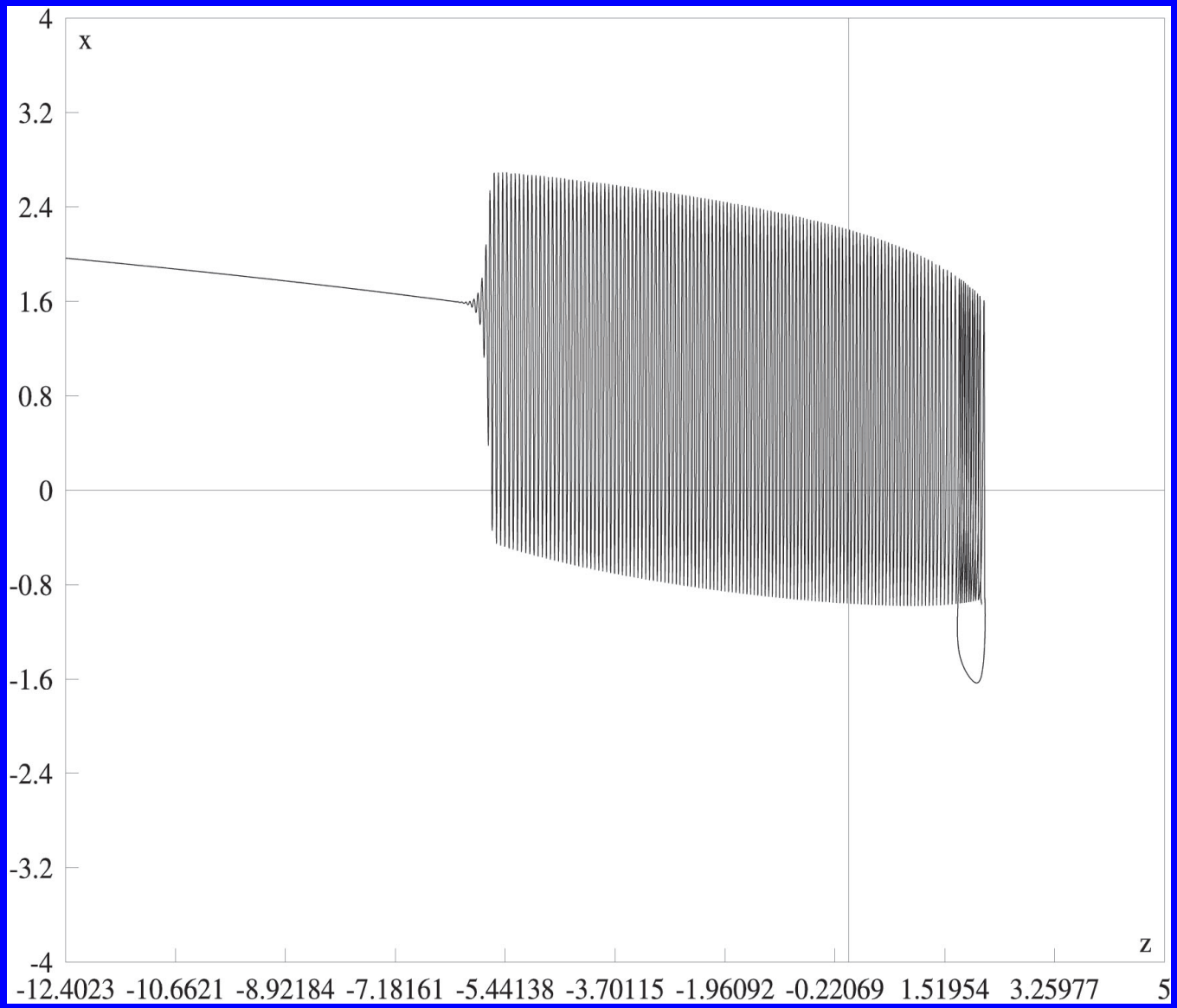

Figure 20: x vs. z in Hindmarsh-Rose model

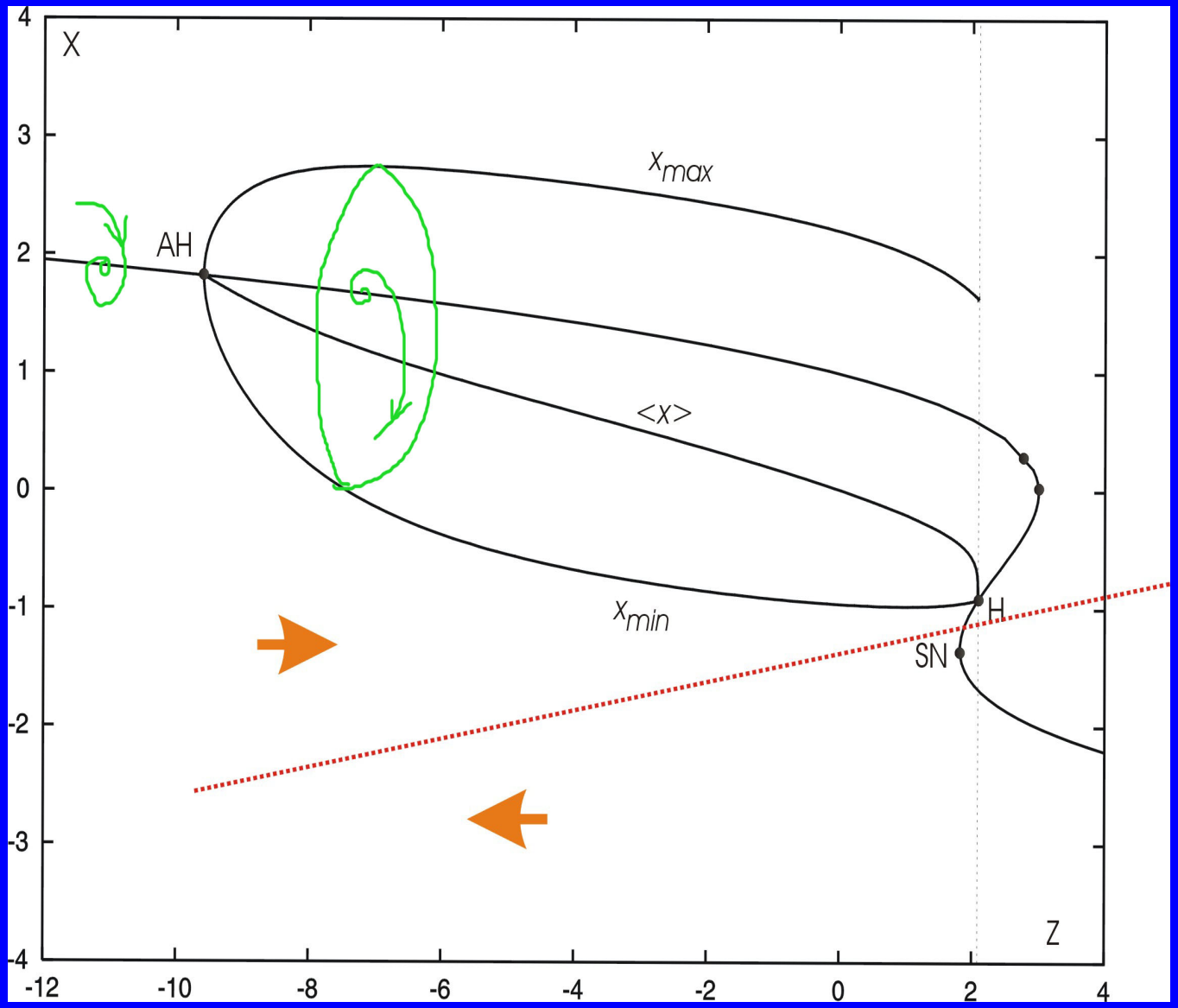

Figure 21: Bifurcation diagram for slow sub-system

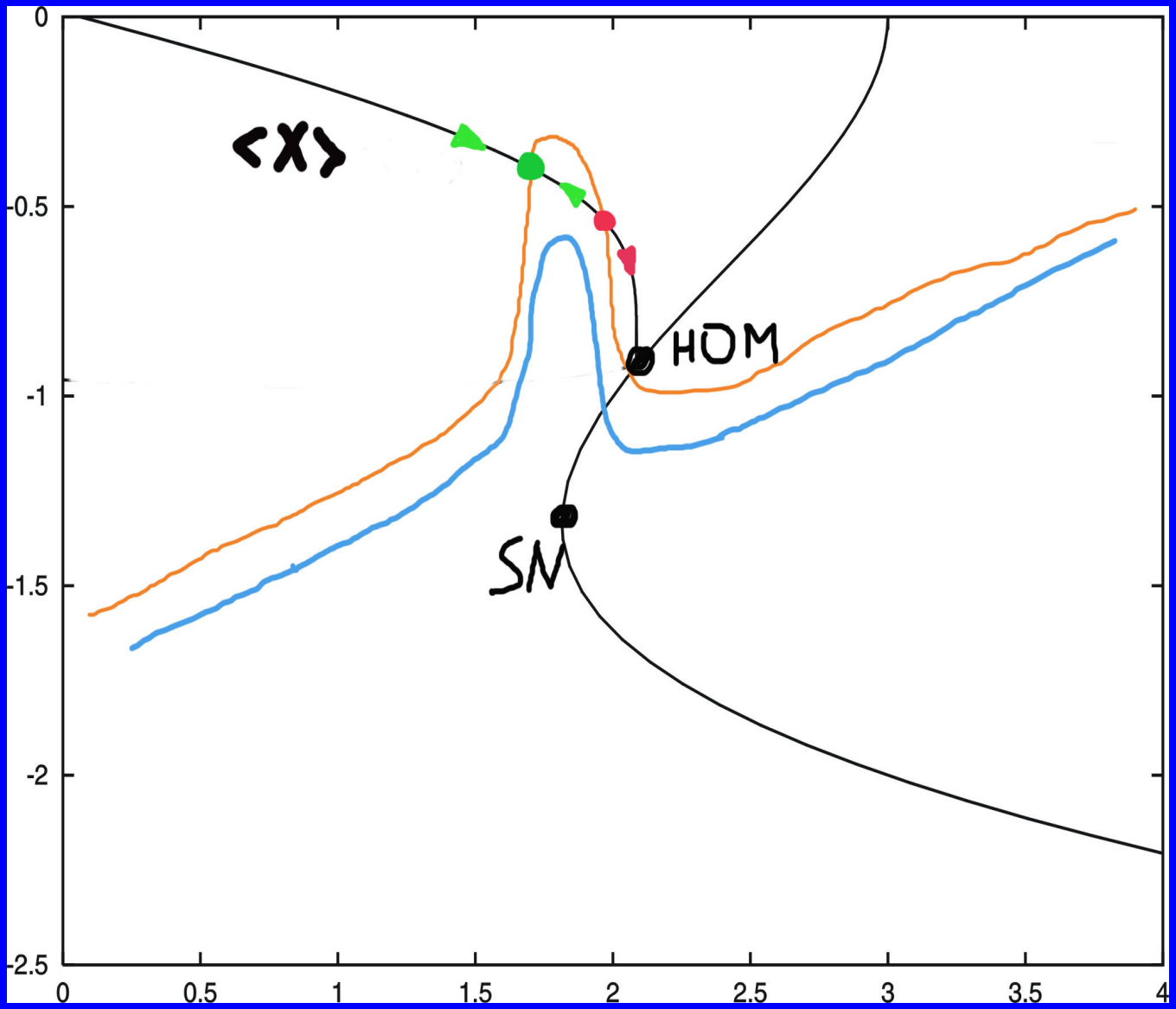

Figure 22: Enlarged fragment of bifurcation diagram

Tuned Hindmarsh-Rose model

$$\begin{cases} \dot{x} = y - z - x^3 + 3x^2 + 5, \\ \dot{y} = -y - 2 - 5x^2, \\ \dot{z} = \varepsilon(2(x + 2.1) - z - \dfrac{A}{(z - 1.93)^2 + 0.003}). \end{cases} \tag{4}$$