Development of Synchronization Theory

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We discuss mathematical problems arising in the classical synchronization theory. The analysis lets us single out the regions of simple and complex dynamics, and give a full description of bifurcations in synchronization problems. At center of consideration are global bifurcations of saddle-node periodic orbits on boundaries of synchronization zones.

I. INTRODUCTION. HOMOCLINIC LOOP UNDER PERIODIC FORCING

These two problems remain fundamental in the classic theory of synchronization: the first one is on the behavior of oscillatory systems driven by a periodical force. The second focuses on the interaction between two coupled oscillatory systems. Both cases give a plethora of dynamical phenomena that exist at different values of control parameters which can either the amplitude and the frequency of the external force, or the coupling strength, respectively.

In terms of the theory of dynamical systems, the problem in question sounds as follows: find a synchronization region in the parameter space that corresponds to the existence of a stable periodic orbit, and describe the ways synchronization is lost after crossing the its boundaries.

Let an autonomous system

$$\begin{aligned} x' &= -\lambda x + f(x, y, \mu), \\ y' &= \gamma y + g(x, y, \mu) \end{aligned}$$

have this stable periodic orbit L_μ that becomes a homoclinic loop to a saddle equilibrium state as $\mu \to 0^+$ such that the saddle value $\sigma = -\lambda + \gamma < 0$. As well known the system shall have a single periodic orbit emerging from the loop for small $\mu > 0$. The period of the new born cycle is of order $|\ln \mu|$. One may wonder what happens as the system is driven periodically by a small force of amplitude of order μ? This problem was studied in a series of papers by Afraimovich and Shilnikov [2–4].

As for the periodically forced system

$$\begin{aligned} x' &= \lambda x + f(x, y, \mu) + \mu p(x, y, t, \mu), \\ y' &= \gamma y + g(x, y, \mu) + \mu q(x, y, t, \mu), \end{aligned} \tag{1}$$

with p and q are 2π-periodic functions in t, we suppose that the stable W^s and unstable W^u manifolds of the saddle fixed point do not cross. Hence, the Poincaré map T_μ is in the form close to the following modelling map:

$$\begin{aligned} \bar{y} &= [y + \mu(1 + f(\theta))]^\nu \\ \bar{\theta} &= \theta + \omega - \tfrac{1}{\gamma} \ln[y + \mu(1 + f(\theta))], \end{aligned} \tag{2}$$

where $\nu = -\frac{\lambda}{\gamma} > 1$, ω is a constant, and $\mu(1+f(\theta))$ is the Melnikov function with $\langle f(\theta) \rangle = 0$. The right-hand side of the second equation is to be evaluated in modulo 2π since θ is an angular variable. The last can be interpreted as the phase difference between the external force and the response of the system. After rescaling $y \to \mu^\nu y$ the map assumes the form

$$\begin{aligned} \bar{y} &= [1 + f(\theta)]^\nu + ... \\ \bar{\theta} &= \theta + \tilde{\omega} + \mathcal{F}(\theta) + ..., \end{aligned} \tag{3}$$

where $\mathcal{F}(\theta) = -\frac{1}{\gamma} \ln[1 + f(\theta)]$. One can see the dynamics is dominated by the second map that de-couples from the first one.

Assertions [4] 1. In the case where

$$\frac{1}{\gamma} \frac{f'(\theta)}{1 + f(\theta)} < 1, \tag{4}$$

the map T_μ has an attracting smooth invariant closed curve of the form $y = h(\theta, \mu)$ that contains ω-limit set of any trajectory in K_μ.

2. Let an interval $I = [\theta_1, \theta_2]$ exist such that either

$$f'(\theta) < 0 \quad \text{everywhere on } I \tag{5}$$

and

$$\frac{1}{\gamma} \ln \frac{1 + f(\theta_1)}{1 + f(\theta_2)} > 2\pi(m + 1), \quad m \geq 2, \tag{6}$$

or

$$\frac{1}{\gamma} \frac{f'(\theta)}{1 + f(\theta)} > 2 \quad \text{everywhere on } I \tag{7}$$

and

$$\frac{1}{\gamma} \ln \frac{1 + f(\theta_2)}{1 + f(\theta_1)} > 2(\theta_2 - \theta_1) + 2\pi(m + 1), \quad m \geq 2. \tag{8}$$

Then, for all sufficiently small $\mu > 0$ the map T_μ will have a hyperbolic set Σ_μ conjugated with the Bernoulli subshift on m symbols.

The meaning of conditions (5) and (7) is that they provide expansion in the θ-variable within the region $\Pi : \theta \in I$ and, therefore, hyperbolicity of the map (2) in the same region (contraction in y is always achieved for all y sufficiently small since $\nu > 1$). Furthermore, if the conditions (6) and (8) are fulfilled, then the image of the region Π overlaps with Π at least m times (see Fig. 1(a)). Hence, we obtain a construction analogous to the Smale horseshoe; then the second assertion above becomes proven by, say, referring to the lemma on a saddle

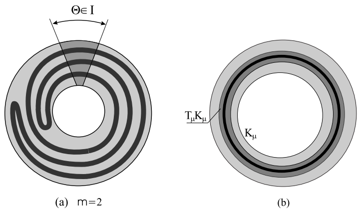

FIG. 1: a) The image of the segment of the region Π overlaps with Π at least m times. b) The image of the annulus K_μ under T_μ has no folds.

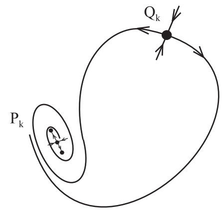

FIG. 3: After a period-doubling on B_k^-, the closure of the unstable manifold of the saddle-fixed point is no longer homeomorphic to circle.

fixed point in a countable product of Banach spaces [10].

In attempt for a comprehensive investigation of the synchronization zones we restrict ourself to the case $f(\theta) = A\sin\theta$ (or $f(\theta) = Ag(\theta)$, where $g(\theta)$ is a function with preset properties). This choice let us build a quite reasonable bifurcation diagram (Figs. 2) in the plane of the parameters $(A, -\ln\mu)$ in the domain $\{0 \le A < 1, 0 < \mu < \mu_0\}$, where μ_0 is sufficiently small. Each such

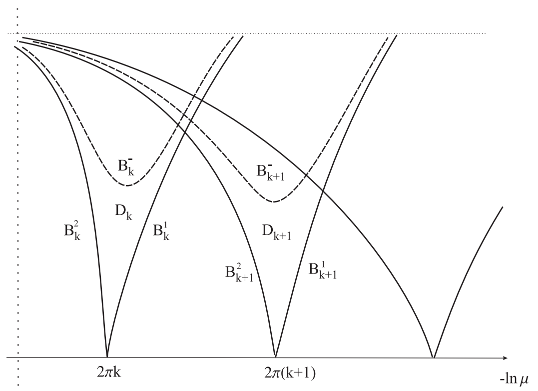

FIG. 2: Overlapping resonant zones.

a region can be shown to adjoin to the axis $-\ln\mu_0$ at a point with the coordinates $(2\pi k, 0)$, where k is a large enough integer. Inside it there co-exist a pair of the fixed points of the Poincaré map such that $\bar{\theta} = \theta + 2\pi k$. Their images in the system (1) are the periodic motions of period $2\pi k$. The borders of a resonant zone D_k are the bifurcation curves B_k^1 and B_k^2 on which the fixed points merge into a single saddle-node. Within the region S_k the closed invariant curve is the unstable manifold W^u of the saddle fixed point Q_k which closes on the stable point P_k.

After crossing B_k^- the invariant curve no longer exists, see Fig. 3.

Another mechanism of breakdown of the invariant circle is due to the onset of homoclinic tangencies produced by the stable and unstable manifolds of the saddle point Q_k. The tangencies occur on the bifurcation curves B_{tk}^1 and B_{tk}^2 where each corresponds to a homoclinic contact to its own component of the set $W^u \backslash Q_k$.

It should be remarked that the synchronization is always *incomplete* in the synchronization zone S_k above the curves B_{tk}^1 and B_{tk}^2. This is due to the likelihood of the presence of other stable periodic orbits of different periods that co-exist along with the orbit L_k corresponding to the stable fixed point P_k of the Poincaré map. However, even if this is not the case and L_k is the only attractor still, the phase difference between L_k and other trajectories from the hyperbolic set nearby the transverse homoclinics to the saddle point Q_k will grow at the asymptotically linear rate, i.e. the phase locking may be broken at least within the transient process.

We should remark too that chaos itself is less important for desynchronization then the presence of homoclinics to the saddle point Q_k. So, for example, in the region $D_k \backslash S_k$ beneath the curves B_{tk}^1 and B_{tk}^2 where Q_k has no homoclinics, the difference in the phase stays always bounded, which means a *relative* synchronization, so to speak. Meanwhile the dynamics can be nonetheless chaotic: for instance, in the region above the curve B_{tk}^0, the fixed point P_k is no longer stable but a saddle with a transverse homoclinic orbit. On the curve B_{tk}^0 its stable and unstable manifolds have a homoclinic tangency of the third class in terminology introduced in [5] which implies particularly the complex dynamics persisting below the curve B_{tk}^0 as well.

Thus, the region D_k corresponding to the existence of the $2\pi k$-periodic orbit, may be decomposed into the zones of complete, incomplete and relative synchronization. The regime of incomplete synchronization, where there are periodic orbits with different rotation numbers, always yields the complex dynamics. Further, in the zone of relative synchronization there is another "non-rotating" type of chaotic behavior. It can be shown that such a tableau of the behavior in the resonance zone D_k is drawn not only for $f(\theta) = A\sin\theta$, but in generic case too for an arbitrary function f.

It is shown in [7] that this homoclinic structure generates the nontrivial hyperbolic set similar to that existing nearby a transverse homoclinic trajectory to a saddle. Upon getting into the region D_k the saddle-node disintegrates becoming a stable node and a saddle, the latter inherits the homoclinic structure, and hence the hyperbolic set persists. Upon exiting D_k the saddle-node dissolves, however a great portion of the hyperbolic set survives [7], i.e. we enter the land of desynchronization ("rotational chaos").

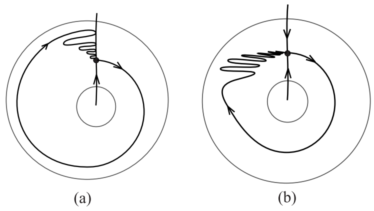

FIG. 4: (a) Homoclinic tangencies involving the unstable and the strongly stable manifolds of a saddle-node. (b) Pre-wiggles of the unstable manifold of the saddle-node.

II. DISAPPEARANCE OF THE SADDLE-NODE

In this section we will analyze a few versions of global saddle-node bifurcations. Let us consider a one-parameter family of $\mathcal{C}^2$-smooth $(n+2)$-dimensional dynamical systems depending smoothly on $\mu \in \mu(-\mu_0;\mu_0)$. Suppose that the following conditions are hold: (1) at $\mu = 0$ the system has a periodic orbit L_0 of the simple saddle-node type. This means that all multipliers besides a single one equal $+1$, lie in the unit circle, and the first Lyapunov coefficient is not zero; (2) all the trajectories in the unstable manifold W^u of L_0 tend to L_0 as $t \to \infty$ and $W^u \cap W^{ss} = \emptyset$, i.e. the returning manifold W^u approaches L_0 from the node region; (3) the family under consideration is transverse to the bifurcational set of systems with a periodic orbit of the saddle-node type. This implies that as μ changes the saddle-node bifurcates: it decouples into a saddle and a node when, say, $\mu < 0$, and does not exist when $\mu > 0$.

According to [12], one may introduce coordinates in a small neighborhood of the orbit L_0 so that the system will assume the following form

$$\begin{aligned} \dot{x} &= \mu + x^2[1 + p(x,\theta,\mu)], \\ \dot{y} &= [A(\mu) + q(x,\theta,y,\mu)]y, \\ \dot{\theta} &= 1, \end{aligned} \tag{9}$$

where the eigenvalues of the matrix A lie in the left open half-plane. Here θ is an angular variable defined modulo of 1. Here, p is a 1-periodic function in θ, whereas q is of period 2. In addition, the indicated coordinates are introduced so that p becomes independent of θ at $\mu = 0$ (the Poincaré map on the center manifold is imbedded into an autonomous flow;).

The saddle-node periodic orbit L_0 is given by equation $(x = 0, y = 0)$ at $\mu = 0$. Its strongly stable manifold W^{ss} is given by $x = 0$. The manifold W^{ss} separates the saddle region (where $x > 0$) of L_0 from the node one where $x < 0$. The manifold $y = 0$ is invariant, this is a center manifold. When $\mu < 0$ it contains two periodic orbits: stable L_1 and saddle L_2, both coalesce in one L_0 at $\mu = 0$. When $\mu > 0$ there are no periodic orbits and a trajectory leaves a small neighborhood of the phantom of the saddle-node.

The analysis of the trajectories near W^u presents interest only when $\mu > 0$ (it is trivial when $\mu \le 0$). When $\mu > 0$ the Poincaré map $T : S_1 \to S_1$ is defined as the superposition of two maps by the orbits of the system: $T_1 : S_1 \to S_0$ followed by $T_0 : S_0 \to S_1$:

$$\begin{aligned} \bar{y} &= g_1(y,\theta,\nu), \\ \bar{\theta} &= m\theta + \nu + f(\theta) + f_1(y,\theta,\nu), \end{aligned} \tag{10}$$

where the functions f_1 and g_1 tend to zero as $\nu \to +\infty$, so do all their derivatives. Thus, we may see that if the fractional part of ν is set fixed, then as its integral part ν tends to infinity, the map T degenerates into the circle map $\tilde{T}$:

$$\bar{\theta} = m\theta + f(\theta) + \nu \mod 1. \tag{11}$$

It becomes evident that the dynamics of the map (10) is dominated by the properties of the map (11).

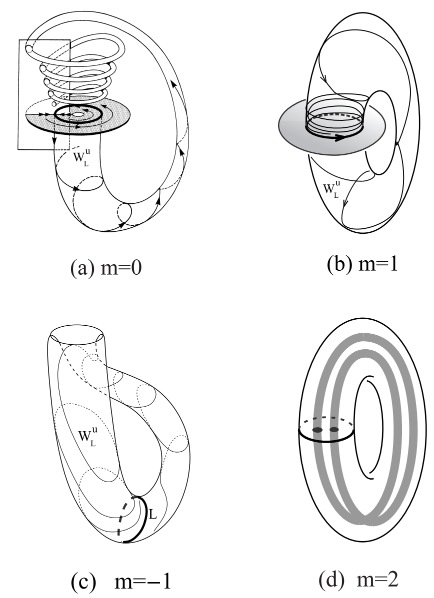

FIG. 5: Case $m = 0$ – the blue sky catastrophe. Cases $m = 1$ and -1: closure of the unstable manifold of the saddle-node periodic orbit is a smooth 2D torus and a Klein bottle, respectively. Case $m = 2$ – the solid-torus is squeezed, doubly expanded and twisted, and inserted back into the original and so on, producing the Wietorius- van Danzig solenoid in the limit

Let us next outline the following two remarkable cases $m = 0$ and $|m| \ge 2$ considered in [8, 11, 12, 14].

Theorem [12, 14]. At $m = 0$ the map T has, for all sufficiently small μ, a single stable fixed point if $|f'(\theta)| < 1$ for all θ.

It comes clear from the theorem that as the orbit L_0 vanishes the stability goes to a new born, single periodic orbit whose length and period both tend to infinity as $\mu \to +0$. This bifurcation is called a blue sky catastrophe. We may refer the reader to the example of the system with the explicitly given right hand side with such catastrophe constructed in [6]. Point out also [8, 9] showing that the blue-sky catastrophe in our setting is typical for singularly perturbed systems with at least two fast variables.

Theorem [11, 14]. Let $|m| \geq 2$ and $|m + f'(\theta)| > 1$ for all θ. Then the map T will have the hyperbolic Smale-William attractor for all small $\mu > 0$.

In these conditions the map T acts similarly to the construction proposed by Smale and Williams. Namely, a solid torus S_0 is mapped into itself in such a way that the limit $\Sigma = \cap_{k \geq 0} T^k S_0$ is a Wietorius- van Danzig solenoid which is locally homeomorphic to the direct product of a Cantor set by an interval.

Theorem [1]. If the limit map $\tilde{T}$ is a diffeomorphism, then for all $\mu > 0$ sufficiently small the map (10) has a closed stable invariant curve attracting all the trajectories of the map.

Introduce a quantity δ defined as

$$\delta = \sup_{\theta_1 < \theta_2} (\theta_1 + f(\theta_1) - \theta_2 - f(\theta_2)).$$

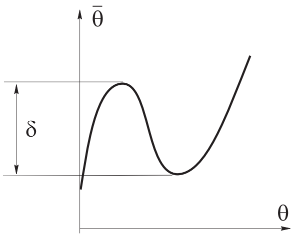

FIG. 6: δ is the absolute value of the difference between certain minimal value of the right-hand side of the map and the preceding maximal one.

It becomes evident that $\delta = 0$ if and only if the map $\tilde{T}$ is a homeomorphism for all ν, i.e. when its graph is an increasing function. If $\delta > 0$, this map is to have at least one point of a maximum as well as one point of a minimum; in essence δ determines the magnitude between the given minimal value of the right-hand side of the map and the preceding maximal one.

Borrowing the terminology introduced in [1] we will refer to the case of $\delta > 1$ as the case of the big lobe.

Theorem [13]. In case of the big lobe the map T has complex dynamics for all $\mu > 0$ sufficiently small.

Theorem [13]. If $\delta > 0$ in the map (11) and all its critical points are of a finite order, then the map T has complex dynamics in the intervals of values of μ which are located arbitrarily close to $\mu = 0$.

Theorem [13]. If $2\delta \max_\theta f''(\theta) < 1$, then arbitrarily close to $\mu = 0$ there are intervals of values of μ where the map T has the trivial dynamics: all trajectories tend to a continuous invariant curve, homeomorphic to a circle, with a finite number of fixed points.

III. ACKNOWLEDGEMENT

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