## 1.2 ROW REDUCTION AND ECHELON FORMS

Our interest in the row reduction algorithm lies mostly in the echelon forms that are created by the algorithm. For practical work, a computer should perform the calculations. However, you need to understand the algorithm so you can learn how to use it for various tasks. Also, unless you take your exams at a computer or with a matrix programmable calculator, you must be able to perform row reduction quickly and accurately by hand.

### STUDY NOTES

The row reduction algorithm applies to any matrix, not just an augmented matrix for a linear system. In many cases, all you need is an echelon form. The reduced echelon form is mainly used when it comes from an augmented matrix and you have to find all the solutions of a linear system.

Strategies for faster and more accurate row reduction:

- Avoid subtraction in a row replacement. It leads to mistakes in arithmetic. Instead, add a negative multiple of one row to another.
- Always enclose each matrix with brackets or large parentheses.
- To save time, combine all row replacement operations that use the same pivot position, and write just one new matrix. Never “clean out” more than one column at a time. (You can combine several scaling operations, or combine several interchanges, if you are careful. But that seldom will be necessary.)
- *Never* combine an interchange with a replacement. In general, don’t combine different types of row operations. This will be particularly important when you evaluate determinants, in Chapters 3 and 5.

How to avoid copying errors:

- Practice neat writing, not too small. Develop proper habits in homework so your work on tests wil1 be accurate, complete and readable.
- Write a matrix row by row. Your eye may be less likely to read from the wrong row if you place the new matrix besides the old one. Arrange your sequence of matrices across the page, rather than down the page. (Some students prefer to place the matrices in columns. Use whichever method seems to work best for you.)
- Try not to let your work flow from one side of a paper to the reverse side.

**Study Tips**: Theorem 2 is a key result for future work. Also, study the procedure in the box following Theorem 2. Failure to write out the system of equations (step 4) is a common source of errors.

## SOLUTIONS TO EXERCISES

**1.** To check whether a matrix is in echelon form ask the questions:

(i) Is every nonzero row above the all-zero rows (if any)?
The matrix (c) fails this test, so it is not in echelon form.

(ii) Are the leading entries in a stair-step pattern, with zeros below each leading entry?
The matrices (a), (b), and (d) all pass test (i) and (ii), so they are in echelon form.

To check whether a matrix in echelon form is actually in reduced echelon form, ask two more questions:

(iii) Is there a 1 in every pivot position?
Matrix (d) fails this test, so it is only in echelon form. Finally, ask:

**Study Tip**: Exercises 5 and 6 ask you to "visualize" echelon forms and write out matrices whose entries are just symbols. Example 2 suggests the form of your "answer," but it does not show you *how to find* the answer. Later, other exercises will ask you to construct other types of examples. If you look at answers from the text, or the *Study Guide* (or another student), before you try to write your own answers, you will lose most of the value of such exercises. The *process* of trying to understand the question and writing an example is important.

**7.** $$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Corresponding system of equations: $$\begin{aligned} x_1 + 3x_2 \qquad &= -5 \\ x_3 &= 3 \end{aligned}$$

The basic variables (corresponding to the pivot positions) are $x_1$ and $x_3$. The remaining variable $x_2$ is free. Solve for the basic variables in terms of the free variable. The general solution is

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

**13.** $$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Corresponding system: $$\begin{aligned} x_1 \qquad\qquad - 3x_5 &= 5 \\ x_2 \qquad - 4x_5 &= 1 \\ x_4 + 9x_5 &= 4 \\ 0 &= 0 \end{aligned}$$

Basic variables: $x_1$, $x_2$, $x_4$; free variables: $x_3$, $x_5$. General solution: $\begin{cases} x_1 = 5 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$

**Note**: A common error in this exercise is to assume that $x_3$ is zero. Another common error is to say *nothing* about $x_3$ and write only $x_1$, $x_2$, $x_4$, and $x_5$ as above. To avoid these mistakes, identify the basic variables first. Any remaining variables are *free*. (This type of computation will arise in Chapter 5.) See also Exercise 8.

**Study Tip**: Be sure to work Exercises 17–20. The experience will help you later. These exercises make nice quiz questions, too.

**19**. $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$. Look first at 8 – 4$h$. If this number is not zero, then the system must be consistent. Also, the solution will be unique because there are no free variables. This is case (b), when $h \neq 2$. Now, if 8 – 4$h$ is zero, that is, if $h$ = 2, there are two possibilities—either $k$ equals 8 or $k$ does not equal 8. If $h$ = 2 and $k$ = 8, the second equation is $0x_2 = 0$. The system is consistent and has a free variable, so the system has infinitely many solutions. This is case (c). When $h$ = 2 and $k \neq 8$, the second equation is $0x_2 = b$, with $b$ nonzero, and the system has no solution. This is case (a).

**21**. **a**. See Theorem 1.

**b**. See the second paragraph of the section.

**c**. Basic variables are defined after equation (4).

**d**. See the beginning of the subsection, “Parametric Descriptions of Solution Sets.” Actually, this question does not consider the case of an inconsistent system. A better true/false statement would be: “If a linear system is consistent, then finding a parametric description of the solution set is the same as *solving* the system.

**e**. The row shown corresponds to the equation $5x_4 = 0$. Could there also be an equation of the form $0x_4 = b$, with $b$ nonzero?

**25**. A full solution is in the text answer section.

**Study Tip**: Notice from Exercise 27 that the question of uniqueness of the solution of a linear system is not influenced by the numbers in the rightmost column of the augmented matrix.

**31**. Yes, a system of linear equations with more equations than unknowns can be consistent. The answer in the text includes an example.

**34**. The data for this exercise comes from one of my students who was working part time for a private wind tunnel company near the University of Maryland. You will need a matrix program to solve this problem. The basic instructions for MATLAB were given in the *Study Guide* notes for Section 1.1. For Maple, Mathematica, the TI-calculators, or the HP-48G calculators, see the respective appendices at the end of this *Study Guide*.

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**A Mathematical Note: "If and only if"**

You need to know what the phrase "if and only if" means. It was used above in Exercise 27, and you will see it again in theorems and in boxed facts. The phrase "if and only if" always appears between two complete statements. Look at Theorem 2, for instance:

$$\left\{\begin{array}{c}\text{A specific}\\\text{linear system}\\\text{is consistent}\end{array}\right\} \text{ if and only if } \left\{\begin{array}{c}\text{the rightmost column of the}\\\text{augmented matrix (of the linear}\\\text{system) is not a pivot column.}\end{array}\right\} \qquad (1)$$

The entire sentence means that the two statements in parentheses are either both true or both false.

Sentence (1) has the general form

$$P \text{ if and only if } Q \qquad (2)$$

where $P$ denotes the first statement and $Q$ denotes the second statement. This sentence says two things:

If statement $P$ is true, then statement $Q$ is also true.
If statement $Q$ is true, then statement $P$ is also true.

A mathematical shorthand for (2) is "$P \Leftrightarrow Q$".

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