1.3 VECTOR EQUATIONS

Do not be deceived by the rather simple beginning of Section 1.3. The important material on Span{$\mathbf{v}_1, \ldots, \mathbf{v}_p$} will take time to digest. Figures 8, 10, and 11 are important, along with Exercises 11–14, 17, 18, 25, and 26. Each of the exercises involves an *existence* question about whether a certain vector equation has a solution. (You don't have to find the solution.) Notice how the same basic question can be asked in several different ways.

STUDY NOTES

Develop the habit of reading the section carefully once or twice before looking at the *Study Guide* and before starting the exercises. (Don't just look at the pictures and examples! Important comments lurk in between.)

In nearly all of the text, a *scalar* is just a real number. By convention, scalars are usually written to the left of vectors, such as 5**v** or *c***v**, rather than **v**5 or **v***c*. To identify vectors in your lecture notes and homework, you can write underlined letters for vectors. (Some students write arrows above the letters, but that takes longer.)

Vectors must be the same size to be added or used in a linear combination. For instance, a vector in $\mathbb{R}^3$ cannot be added to a vector in $\mathbb{R}^2$.

SOLUTIONS TO EXERCISES

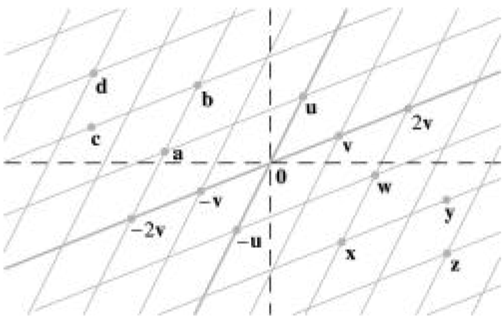
1. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+(-3)\\2+(-1)\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$.

Using the definitions carefully,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+(-2)\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}(-2)(-3)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}\text{, or, more quickly,}$$

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}-2\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}.\ \text{The intermediate step is often not written.}$$

7. See the figure below. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbb{R}^2$ can be written as a linear combination of **u** and **v**. To write a vector **a** as a linear combination of **u** and **v**, imagine walking from the origin to **a** along the grid "streets" and keep track of how many "blocks" you travel in the **u**-direction and how many in the **v**-direction.



These are UNCORRECTED page proofs.

a. To reach **a** from the origin, you might travel 1 unit in the **u**-direction and -2 units in the **v**-direction (that is, 2 units in the negative **v**-direction). Hence $\mathbf{a} = \mathbf{u} - 2\mathbf{v}$.

b. To reach **b** from the origin, travel 2 units in the **u**-direction and -2 units in the **v**-direction. So $\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$. Or, use the fact that **b** is 1 unit in the **u**-direction from **a**, so that

$$\mathbf{b} = \mathbf{a} + \mathbf{u} = (\mathbf{u} - 2\mathbf{v}) + \mathbf{u} = 2\mathbf{u} - 2\mathbf{v}$$

c. The vector **c** is -1.5 units from **b** in the **v**-direction, so

$$\mathbf{c} = \mathbf{b} - 1.5\mathbf{v} = (2\mathbf{u} - 2\mathbf{v}) - 1.5\mathbf{v} = 2\mathbf{u} - 3.5\mathbf{v}$$

d. The "map" suggests that you can reach **d** if you travel 3 units in the **u**-direction and -4 units in the **v**-direction. If you prefer to stay on the paths displayed on the map, you might travel from the origin to $-3\mathbf{v}$, then move 3 units in the **u**-direction, and finally move -1 unit in the **v**-direction. So

$$\mathbf{d} = -3\mathbf{v} + 3\mathbf{u} - \mathbf{v} = 3\mathbf{u} - 4\mathbf{v}$$

Another solution is

$$\mathbf{d} = \mathbf{b} - 2\mathbf{v} + \mathbf{u} = (2\mathbf{u} - 2\mathbf{v}) - 2\mathbf{v} + \mathbf{u} = 3\mathbf{u} - 4\mathbf{v}$$

9. Here are the intermediate calculations, which usually are not displayed. Check with your instructor whether you need to "show work" on a problem such as this.

$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0, \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} x_2 + 5x_3 \\ 4x_1 + 6x_2 - x_3 \\ -x_1 + 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 4x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 6x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -x_3 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_1\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Helpful Hint: As you work Exercises 11–14, circle the pivots in an echelon form of an appropriate matrix. This will help you visualize the cases when a vector either is or is not a linear combination of other vectors.

13. Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. To determine if $\mathbf{b}$ is a linear combination of these columns, use the boxed fact on page 34. Row reduce the augmented matrix until you reach echelon form:

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The system for this augmented matrix is inconsistent, so $\mathbf{b}$ is *not* a linear combination of the columns of A.

19. By inspection, $\mathbf{v}_2 = (3/2)\mathbf{v}_1$. Any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is actually just a multiple of $\mathbf{v}_1$. For instance,

$$a\mathbf{v}_1 + b\mathbf{v}_2 = a\mathbf{v}_1 + b(3/2)\mathbf{v}_1 = (a + 3b/2)\mathbf{v}_1$$

So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of points on the line through $\mathbf{v}_1$ and $\mathbf{0}$.

Warning: Although Figures 8 and 11 provide the most common ways to view Span$\{\mathbf{u}, \mathbf{v}\}$, don't forget Exercise 19, which shows that in a special case, Span$\{\mathbf{u}, \mathbf{v}\}$ can be just a line through the origin. In fact, Span$\{\mathbf{u}, \mathbf{v}\}$ can also be just the origin itself. How?

21. Let $\mathbf{y} = \begin{bmatrix} h \\ k \end{bmatrix}$. Then $[\mathbf{u} \quad \mathbf{v} \quad \mathbf{y}] = \begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & h \\ 0 & 2 & k + h/2 \end{bmatrix}$. This augmented matrix corresponds to a consistent system for all h and k. So $\mathbf{y}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

23. **a**. The alternative notation for a (column) vector is discussed after Example 1.

b. Plot the points to check the assertion. Or, see the statement preceding Example 3.

c. See the line displayed just before Example 4.

d. See the box that discusses the matrix in (5).

e. Read the geometric description of Span$\{\mathbf{u}, \mathbf{v}\}$ very carefully.

Study Tip: I urge my own students to work by themselves on the true/false questions and then meet together in groups of two or three, to compare and discuss their answers.

25. a. There are only three vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and $\mathbf{b}$ is not one of them.

b. There are infinitely many vectors in W = Span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. To determine if $\mathbf{b}$ is in W, use the method of Exercise 13.

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{matrix} \end{array} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

The system for this augmented matrix is consistent, so **b** is in W.

c. $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$. See the discussion following the definition of Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

31. **a**. The center of mass is $\frac{1}{3}\left(1\cdot\begin{bmatrix}0\\1\end{bmatrix}+1\cdot\begin{bmatrix}8\\1\end{bmatrix}+1\cdot\begin{bmatrix}2\\4\end{bmatrix}\right)=\begin{bmatrix}10/3\\2\end{bmatrix}$.

b. The total mass of the new system is 9 grams. The three masses added, w_1, w_2, and w_3, satisfy the equation

$$\frac{1}{9}\left((w_1+1)\cdot\begin{bmatrix}0\\1\end{bmatrix}+(w_2+1)\cdot\begin{bmatrix}8\\1\end{bmatrix}+(w_3+1)\cdot\begin{bmatrix}2\\4\end{bmatrix}\right)=\begin{bmatrix}2\\2\end{bmatrix}$$

which can be rearranged to

$$(w_1+1)\cdot\begin{bmatrix}0\\1\end{bmatrix}+(w_2+1)\cdot\begin{bmatrix}8\\1\end{bmatrix}+(w_3+1)\cdot\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}18\\18\end{bmatrix}$$

and

$$w_1\cdot\begin{bmatrix}0\\1\end{bmatrix}+w_2\cdot\begin{bmatrix}8\\1\end{bmatrix}+w_3\cdot\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}8\\12\end{bmatrix}$$

The condition $w_1 + w_2 + w_3 = 6$ and the vector equation above combine to produce a system of three equations whose augmented matrix is shown below, along with a sequence of row operations:

$$\begin{bmatrix}1&1&1&6\\0&8&2&8\\1&1&4&12\end{bmatrix}\sim\begin{bmatrix}1&1&1&6\\0&8&2&8\\0&0&3&6\end{bmatrix}\sim\begin{bmatrix}1&1&1&6\\0&8&2&8\\0&0&1&2\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&0&4\\0&8&0&4\\0&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&3.5\\0&8&0&4\\0&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&3.5\\0&1&0&.5\\0&0&1&2\end{bmatrix}$$

Answer: Add 3.5 g at (0, 1), add .5 g at (8, 1), and add 2 g at (2, 4).

33. **a**. For $j = 1, \ldots, n$, the jth entry of $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ is $(u_j + v_j) + w_j$. By associativity of addition in $\mathbb{R}$, this entry equals $u_j + (v_j + w_j)$, which is the jth entry of $\mathbf{u} + (\mathbf{v} + \mathbf{w})$. By definition of equality of vectors, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

b. For any scalar c, the jth entry of $c(\mathbf{u} + \mathbf{v})$ is $c(u_j + v_j)$, and the jth entry of $c\mathbf{u} + c\mathbf{v}$ is $cu_j + cv_j$ (by definition of scalar multiplication and vector addition). These entries are equal, by a distributive law in $\mathbb{R}$. So $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

MATLAB Constructing a Matrix

To access the data for Section 1.3, give the command **cls3**. The data for Exercise 25, for example, consists of a matrix A, its columns **a**l, **a**2, **a**3, and the vector **b**. The command **M** = **[al a2 a3 b]** creates a matrix using the vectors as its columns. The same matrix is created by the command **M** = **[A b]**.

Each time you want data for a new exercise in Section 1.3, you need the command **cls3**. After the first exercise, you can use the up-arrow (↑). This will make MATLAB scroll back through your old commands. You may be able to find “c2sl” faster than you can retype it. Press <Enter> to reuse the command.

Exercises 11–14, 25–28, and 31 can be solved using the commands **replace**, **swap**, and (occasionally) **scale**, described on page 1-6.