1.5 SOLUTION SETS OF LINEAR SYSTEMS

Many of the concepts and computations in linear algebra involve sets of vectors which are visualized geometrically as lines and planes. The most important examples of such sets are the solution sets of linear systems.

KEY IDEAS

Visualize the solution set of a homogeneous equation $A\mathbf{x} = \mathbf{0}$ as:

- the single point $\mathbf{0}$, when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution,
- a line through $\mathbf{0}$, when $\mathrm{A}\mathbf{x} = \mathbf{0}$ has one free variable,
- a plane through $\mathbf{0}$, when $A\mathbf{x} = \mathbf{0}$ has two free variables.
  (For more than two free variables, also use a plane through $\mathbf{0}$.)

For **b**   **0**, visualize the solution set of $A\mathbf{x} = \mathbf{b}$ as:

- empty, if **b** is not a linear combination of the columns of A,
- one nonzero point (vector), when $A\mathbf{x} = \mathbf{b}$ has a unique solution,
- a line not through **0**, when $A\mathbf{x} = \mathbf{b}$ is consistent and has one free variable,
- a plane not through **0**, when $A\mathbf{x} = \mathbf{b}$ is consistent and has two or more free variables.

The solution set of $A\mathbf{x} = \mathbf{b}$ is said to be described *implicitly*, because the equation is a condition an **x** must satisfy in order to be in the set, yet the equation does not show how to find such an **x**. When the solution set of $A\mathbf{x} = \mathbf{0}$ is written as Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, the set is said to be described *explicitly*; each element in the set is produced by forming a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}p$.

A common explicit description of a set is an equation in *parametric vector form*. Examples are:

$\mathbf{x} = t\mathbf{v},$	a line through **0** in the direction of **v**,
$\mathbf{x} = \mathbf{p} + t\mathbf{v},$	a line through **p** in the direction of **v**,
$\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v},$	a plane through **0**, **u** and **v**,
$\mathbf{x} = \mathbf{p} + x_2\mathbf{u} + x_3\mathbf{v},$	a plane through **p** parallel to the plane whose equation is $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$.

An equation in parametric vector form describes a set explicitly because the equation shows how to produce each **x** in the set.

To *solve* an equation $A\mathbf{x} = \mathbf{b}$ means to find an explicit description of the solution set. If the system is inconsistent, the solution set is empty. Otherwise, the description of all solutions can be written in parametric vector form, in which the parameters are the free variables from the system. *Important*: The number of free variables in $A\mathbf{x} = \mathbf{b}$ depends only on A, not on **b**.

Theorem 6 and the paragraph following it are important. They describe how the solutions of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ are related when solution set of $A\mathbf{x} = \mathbf{b}$ is nonempty. See Figs. 5 and 6. Key exercises: 5–16, 29–32, 37.

SOLUTIONS TO EXERCISES

1. Reduce the augmented matrix to echelon form and circle the pivot positions. If a column of the *coefficient* matrix is not a pivot column, the corresponding variable is free and the system of equations has a nontrivial solution. Otherwise, the system has *only* the trivial solution.

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variable x_3 is free, so the system has a nontrivial solution.

7. Always use the *reduced* echelon form of an augmented matrix to find the solutions of a system. See the text's discussion of back substitution on pages 22–23.

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 \\ 0 & 1 & -4 & 5 \end{bmatrix}, \quad \begin{aligned} x_1 \quad + 9x_3 &= -8 \\ x_2 - 4x_3 &= 5 \end{aligned}$$

If you wrote something like the system above, then you made a common mistake. The matrix in the text problem is a coefficient matrix, not an augmented matrix. You should row reduce $[A \quad \mathbf{0}]$. The correct system of equations is

$$\begin{aligned} x_1 \quad + 9x_3 - 8x_4 &= 0 \\ x_2 - 4x_3 + 5x_4 &= 0 \end{aligned}$$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = -9x_3 + 8x_4$, and $x_2 = 4x_3 - 5x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

The solution set is the same as Span$\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = (-9, 4, 1, 0)$ and $\mathbf{v} = (8, -5, 0, 1)$. Originally, the solution set was described implicitly, by a set of equations. Now the solutions set is described explicitly, in parametric vector form.

11.

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 4x_2 \qquad\qquad + 5x_6 &= 0 \\ x_3 \qquad - x_6 &= 0 \\ x_5 - 4x_6 &= 0 \\ 0 &= 0 \end{aligned}.$$

These are UNCORRECTED page proofs.

Some students are not sure what to do with x_4. Some ignore it; others set it equal to zero. In fact, x_4 is free; there is no constraint on x_4 at all. The basic variables are x_1, x_3, and x_5. The remaining variables are free. So, $x_1 = 4x_2 - 5x_6$, $x_3 = x_6$, and $x_5 = 4x_6$, with x_2, x_4, and x_6 free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_6 \\ 0 \\ x_6 \\ 0 \\ 4x_6 \\ x_6 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}} + x_6 \underbrace{\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}}_{\mathbf{w}}$$

The solution set is the same as Span{**u**, **v**, **w**}.

Study Tip: When solving a system, identify (and perhaps circle) the basic variables. All other variables are free.

13. To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 + 4x_3 \\ -2 - 7x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -7x_3 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}}_{\mathbf{q}} = \mathbf{p} + x_3\mathbf{q}$$

Geometrically, the solution set is the line through $\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ parallel to $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$.

Checkpoint: Let A be a 2 3 2 matrix. Answer True or False: If the solution set of $A\mathbf{x} = \mathbf{0}$ is a line through the origin in R^2 and if $\mathbf{b} \neq \mathbf{0}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is a line not through the origin.

19. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \text{ or } \begin{cases} x_1 = -2 - 5t \\ x_2 = 3t \end{cases}$$

23. **a**. See the first paragraph of the subsection titled "Homogeneous Linear Systems."

b. See the first two sentences of the subsection titled "Parametric Vector Form."

c. See the box before Example 1.

d. See the paragraph that precedes Fig. 5.

e. See Theorem 6.

25. Suppose **p** satisfies $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{p} = \mathbf{b}$. Theorem 6 says that the solution set of $A\mathbf{x} = \mathbf{b}$ equals the set $S = \{\mathbf{w} : \mathbf{w} = \mathbf{p} + \mathbf{v}_h$ for some $\mathbf{v}_h$ such that $A\mathbf{v}_h = 0\}$. There are two things to prove: (a) every vector in S satisfies $A\mathbf{x} = \mathbf{b}$, (b) every vector that satisfies $A\mathbf{x} = \mathbf{b}$ is in S.

a. Let **w** have the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $A\mathbf{v}_h = \mathbf{0}$. Then

$$A\mathbf{w} = A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h \qquad \text{By Theorem 5(a) in Section 1.4}$$
$$= \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So every vector of the form $\mathbf{p} + \mathbf{v}_h$ satisfies $A\mathbf{x} = \mathbf{b}$.

b. Now let **w** be any solution of $A\mathbf{x} = \mathbf{b}$, and set $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Then

$$A\mathbf{v}_h = A(\mathbf{w} - \mathbf{p}) = A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

So $\mathbf{v}_\mathbf{h}$ satisfies $A\mathbf{x} = \mathbf{0}$. Thus every solution of $A\mathbf{x} = \mathbf{b}$ has the form

$\mathbf{w} = \mathbf{p} + \mathbf{v}_h$.

31. A is a 3×2 matrix with two pivot positions.

a. Since A has a pivot position in each column, each variable in $A\mathbf{x} = \mathbf{0}$ is a basic variable. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence no nontrivial solution.

b. With two pivot positions and three rows, A cannot have a pivot in every row. So the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible **b** (in $\mathbb{R}^3$), by Theorem 4 in Section 1.4.

37. If you worked on the Checkpoint when you first saw it, you should be ready for this exercise. Since the solution set of $A\mathbf{x} = \mathbf{0}$ contains the point (4, 1), the vector $\mathbf{x} = (4, 1)$ satisfies $A\mathbf{x} = \mathbf{0}$. Write this equation as a vector equation, using $\mathbf{a}_1$ and $\mathbf{a}_2$ for the columns of A:

$$4 \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_2 = \mathbf{0}$$

Then $\mathbf{a}_2 = -4\mathbf{a}_1$. So choose any nonzero vector for the first column of A and multiply that column by -4 to get the second column of A. For example, set $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$.

Finally, the only way the solution set of $A\mathbf{x} = \mathbf{b}$ could *not* be parallel to the line through (4, 1) and the origin is for the solution set of $A\mathbf{x} = \mathbf{b}$ to be *empty*. (Theorem 6 applies only to the case when the equation $A\mathbf{x} = \mathbf{b}$ has a nonempty solution set.) For **b**, take any vector that is *not* a multiple of the columns of A.

Answer to Checkpoint: False. The solution set could be empty. In this case, the solution set of $A\mathbf{x} = \mathbf{b}$ is not produced by translating the (nonempty) solution set of $A\mathbf{x} = \mathbf{0}$. See the Warning after Theorem 6.

MATLAB Zero Matrices

The command **`zeros(m,n)`** creates an m 3 n matrix of zeros. When solving an equation $A\mathbf{x} = \mathbf{0}$, create an augmented matrix:

```
M = [A zeros(m, 1)]
```
m is the number of rows in *A*.

Then use **`gauss, bgauss,`** and **`scale`** to row reduce M completely.