1.7 LINEAR INDEPENDENCE

This section is as important as Section 1.4 and should be studied just as carefully. Full understanding of the concepts will take time, so get started on the section now.

KEY IDEAS

Figures 1 and 2, along with Theorem 7, will help you understand the nature of a linearly dependent set. (Fig. 2 applies only when **u** and **v** are independent.) But you must also learn the *definitions* of linear dependence and linear independence, word for word! Many theoretical problems involving a linearly dependent set are treated by the definition, because it provides an equation (the dependence equation) with which to work. (See the proof of Theorem 7.)

The box before Example 2 contains a very useful fact. Any time you need to study the linear independence of a set of p vectors in $\mathbb{R}^n$, you can always form an n 3 p matrix A with those vectors as columns and then study the matrix equation $A\mathbf{x} = \mathbf{0}$. This is not the only method, however. Stay alert for three special situations:

- A set of two vectors. Always check this by inspection; don't waste time on row reduction of $[A \quad \mathbf{0}]$. The set is linearly independent if neither of the vectors is a multiple of the other. (For brevity, I sometimes say that "the vectors are not multiples.") See Example 3.
- A set that contains too many vectors, that is, more vectors than entries in the vectors; the columns of a short, fat matrix. Theorem 8.
- A set that contains the zero vector. Theorem 9.

The most common mistake students make when checking a set of three or more vectors for independence is to think they only have to verify that no vector is a multiple of one of the other vectors. Wrong! Study Example 5 and Figure 4.

Key exercises are 9–20 and 23–28, and 30. Try Exercise 35, even if it is not assigned. Think carefully, and write your answer before checking the answer section.

SOLUTIONS TO EXERCISES

1. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

Warning: Whenever you study a homogeneous equation, you may be tempted to omit the augmented column of zeros because it never changes under row operations. I urge you to keep the zeros, to avoid possibly misinterpreting your own calculations. In Exercise 1, if you wrote

$$\begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

you might conclude that "the system is inconsistent" and then go on to make some crazy statement about linear dependence or independence. Don't laugh. I have seen this happen on exams. A more common error occurs in a problem like Exercise 7. In that exercise, if you write

$$\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 11 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

you might concluded that "the system has a unique solution" and the vectors are linearly independent. However, the four columns are actually linearly dependent. In both cases, the error is to misinterpret your matrix as an augmented matrix.

7. Study the equation $A\mathbf{x} = \mathbf{0}$. Some people may start with the method of Example 2:

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 11 & -5 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & -6 & 0 \end{bmatrix}$$

But this is a waste of time. There are only 3 rows, so there are at most three pivot positions. Hence, at least one of the four variables must be free. So the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.

Warning: Exercise 9 and Practice Problem 3 emphasize that to check whether a set such as $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, it is *not* wise to check instead whether $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

13. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

Checkpoint: What is wrong with the following statement?

The vectors $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ -4 \end{bmatrix}$ are linearly dependent "because there is a free variable," or "because there are more variables than equations."

15. The set $\left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix} \right\}$ is obviously linearly dependent, by Theorem 8, because there are more vectors (4) than entries in the vectors. On a test, you probably will not have to know the theorem number. Check with your instructor.

19. The set is linearly independent because neither vector is a multiple of the other vector. [Two of the entries in the first vector are –4 times the corresponding entry in the second vector. But this multiple does not work for the third entries.]

21. **a**. See the box before Example 2.

b. See the warning after Theorem 7.

c. See Fig. 3, after Theorem 8.

d. See the remark following Example 4.

25. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

31. Think of $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$. The text points out that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$. Rewrite this as $\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, 1, -1)$.

33. The text uses Theorem 7 to conclude that $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is linearly dependent. Another argument is to rewrite the equation $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$ as $2\mathbf{v}_1 + 1\mathbf{v}_2 + (-1)\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$. This is a linear dependence relation. Some students think of this argument rather than Theorem 7. Did you? (I hope you did not read the answer before trying this problem.

37. True. The text gives a complete answer.

39. If for all **b** the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution, then take $\mathbf{b} = \mathbf{0}$, and conclude that the equation $A\mathbf{x} = \mathbf{0}$ has at most one solution. Then the trivial solution is the only solution, and so the columns of A are linearly independent.

43. **[M]** Make **v** any one of the columns of A that is not in B and row reduce the augmented matrix $[B \quad \mathbf{v}]$. The calculations will show that the equation $B\mathbf{x} = \mathbf{v}$ is consistent, which means that **v** is a linear combination of the columns of B. Thus, each column of A that is not a column of B is in the set spanned by the columns of B.

Answer to Checkpoint: The set of four vectors contains only vectors, no variables of any kind, and no equations. It makes no sense to talk about the variables in a set of vectors. Variables appear in an equation. One cannot assume that the writer of the statement has any idea of the appropriate equation. If you want to give an explanation involving variables, then you must specify the equation. One correct answer is: the vectors are linearly dependent because the equation $x_1\begin{bmatrix} 3 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 3 \end{bmatrix} + x_4\begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ necessarily has a free variable.

Mastering Linear Algebra Concepts: Linear Independence

In Section 1.4 of this *Guide*, I described how to begin forming a mental image of the concept of a spanning set. The same technique works for linear independence. The goal is to merge all the ideas you find regarding linear independence into a single mental image, with each part immediately available in your mind for use as needed. Start now to organize on paper your understanding of linear independence/dependence, using the following list as a guide. In each case, write information that you think will be helpful. (Definitions and theorems should be copied word-for-word.)

- definitions of linear independence and dependence
- equivalent descriptions — Theorem 7
- geometric interpretations — Figs. 1, 2, 4
- special cases — Theorems 8, 9, box on p. 67, Examples 3, 5, 6
- examples and "counterexamples" — Figs. 1, 2, 3, 4, Exercises 9–20, 33–38
- algorithms or typical computations — Examples 1, 2, Exercises 1–8
- connections with other concepts — Box on p. 66, Examples 2, 4, Exercises 27, 30, 39

As you work on your notes, be careful to use terminology correctly. For instance, the term "linearly independent" may be applied to a set of vectors, but it *never* is applied to a matrix or to an equation. The *columns* of a matrix may be linearly independent, but it is meaningless to refer to a linearly independent matrix. Similarly, *solutions* of a system of linear equations may be linearly independent, but the term "linearly independent equations" has never been defined. Finally, a set of vectors or a matrix cannot have a "nontrivial solution". Only equations have solutions.