1.4 THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

The ideas, boxed statements, and theorems in this section are absolutely fundamental for the rest of the text, so you should read the section extremely carefully.

KEY IDEAS

The definition of $A\mathbf{x}$ as a linear combination of the columns of A will be used often. You should learn the definition in *words* as well as symbols. *Note*: It is not wrong to write a scalar on the *right* side of a vector and write $A\mathbf{x}$ as $\mathbf{a}_1x_1 + \cdots + \mathbf{a}_nx_n$, but the text follows the usual practice of writing a scalar on the *left* side of a vector.

You need to understand *why* Theorem 4 is true. That may take some time and effort. Example 3 should help, along with the proof. Theorem 4(d) can be restated as “The reduced echelon form of A has no row of zeros.”

The phrase *logically equivalent* is explained in the statement of Theorem 4. This phrase is used with several statements in the same way that *if and only if* (or the symbol $\Leftrightarrow$) is used between two statements. (See the Mathematical Note at the end of Section 1.2 in this *Guide*.)

Saying that statements (a), (b), (c), and (d) are logically equivalent means the same thing as saying that (a) ⇔ (b), (b) ⇔ (c), and (c) ⇔ (d).

Key exercises are 1–20, 27, 28, 31 and 32. Think about 31 and 32, even if they are not assigned, because they introduce ideas you will need soon. (Don't check the solution of Exercise 31 until you have written your own answer.)

Checkpoint 1: True or False? If an augmented matrix $[A \quad \mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Note: You should work a checkpoint problem when you first see it, provided that you have already read the text at least once. Always *write* your answer before comparing it with the one I have written. The checkpoint answer will be at the end of the solutions for this section.

SOLUTIONS TO EXERCISES

1. The text has the solution. Exercises 1–12 are designed to help you learn Theorem 3 and the definition of $A\mathbf{x}$. If a problem involves vectors–say, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$– you can place the vectors into a matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$, if that is helpful. If a problem involves a matrix A, you can give names to the columns of A–say, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$–and reformulate a matrix equation as a vector equation. If a problem leads to a system of linear equations, you may regard it as either a vector equation or a matrix equation, whichever is most useful.

7. The left side of the equation is a linear combination of three vectors. Write the matrix A whose columns are those three vectors, and create a variable vector $\mathbf{x}$ with three entries:

$$A = \left[\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus the equation $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

Warning: Be careful to distinguish between the *matrix equation* $A\mathbf{x} = \mathbf{b}$ and the *augmented matrix* $[\mathbf{a}_1 \ \cdots \ \mathbf{a}_p \ \ \mathbf{b}]$, which is used in Theorem 3 to refer to a system of linear equations having this augmented matrix. Thus, the answer to Exercise 7 is *not* the augmented matrix at the right:

$$\begin{bmatrix} 4 & -5 & 7 & 6 \\ -1 & 3 & -8 & -8 \\ 7 & -5 & 0 & 0 \\ -4 & 1 & 2 & -7 \end{bmatrix}$$

13. The vector $\mathbf{u}$ is in the plane spanned by the columns of A if and only if $\mathbf{u}$ is a linear combination of the columns of A. This happens if and only if the equation $A\mathbf{x} = \mathbf{u}$ has a solution. (See the box preceding Example 3 in Section 1.4.) To study this equation, reduce the augmented matrix $[A \ \ \mathbf{u}]$:

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 1 & 4 \\ 0 & \textcircled{8} & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has a solution, so $\mathbf{u}$ is in the plane spanned by the columns of A.

Study Tip: Exercises 17–20 require written explanations as well as calculations. For instance, your calculation for Exercise 17 might show the row reduction

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 0 & 3 \\ 0 & \textcircled{2} & -1 & 4 \\ 0 & 0 & 0 & \textcircled{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

After this, it is not enough to write "No, by Theorem 4." Instead, you should show that you know *why* Theorem 4 is relevant. For instance, you might write:

> The matrix A does *not* have a pivot in every row. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ does *not* have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$.

On a test, you probably would not have to know the theorem number. It might be enough to say "By a theorem," instead of "By Theorem 4." (Check with your instructor.)

19. The work in Exercise 17 shows that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for each $\mathbf{b}$. That is, statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Since statement (b) is false, not all vectors in $\mathbb{R}^4$ can be written as a linear combination of the columns of A. Since statement (c) is false, the columns of A do *not* span $\mathbb{R}^4$.

Checkpoint 2 Given $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ as in Exercise 21, find a specific vector in $\mathbb{R}^4$ that is not in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (If necessary, reread Example 3.)

23. **a**. See the paragraph following equation (3). **b**. See the box before Example 3.
c. See the warning following Theorem 4. **d**. See Example 4.
e. See Theorem 4. **f**. See Theorem 4.

25. By definition, the matrix-vector product on the left is a linear combination of the columns of the matrix, in this case using weights –3, –1, and 2. So $c_1 = -3$, $c_2 = -1$, and $c_3 = 2$.

29. Start with any 3 3 3 matrix *B* in echelon form that has three pivot positions. Perform a row operation (a row interchange or a row replacement) that creates a matrix *A* that is *not* in echelon form. Then *A* has the desired property. The justification is given by row reducing *A* to *B*, in order to display the pivot positions. Since *A* has a pivot position in every row, the columns of *A* span $\mathbb{R}^3$, by Theorem 4.

31. A 3 3 2 matrix has three rows and two columns. With only two columns, *A* can have at most two pivot columns, and so *A* has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$.

33. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of *A* is a pivot column. So the reduced echelon form of *A* must be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

37. **[M]** The original matrix has no pivot in the fourth row, so its columns do not span $\mathbb{R}^4$, by Theorem 4.

Helpful Hint: For Exercises 41 and 42, use a matrix program to obtain an echelon form of the matrix. Try covering various columns of this matrix, one at a time, and ask yourself if the columns of the resulting matrix span $\mathbb{R}^4$. If you can delete one column, can you delete a second column? Why or why not?

The analysis here depends on the following idea, which is fairly obvious but is not explicitly mentioned in the text. When a row operation is performed on a matrix *A*, the calculations for each new entry depend only on the other entries in the *same column*. If a column of *A* is removed, forming a new matrix, the absence of this column has no affect on any row-operation calculations for entries in the other columns of *A*. (The absence of a column might affect the particular *choice* of row operations performed for some purpose, but that is not relevant.)

Answers to Checkpoints:

1. False. See the Warning after Theorem 4. If you missed this, you are not studying the text properly. You should read the text thoroughly *before* you look at the *Study Guide* and before you work on the exercises.

2. Let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$. Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$ to determine values of $b_1, \ldots, b_4$ that make the equation *in*consistent.

$$\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 0 & b_2 \\ -1 & 0 & 0 & b_3 \\ 0 & -1 & -1 & b_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 + b_1 \\ 0 & -1 & -1 & b_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 + b_1 \\ 0 & 0 & -1 & b_4 + b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 + b_1 \\ 0 & 0 & 0 & b_4 + b_2 + b_3 + b_1 \end{bmatrix}$$

Take $\mathbf{b} = (1, 1, 0, 0)$, for example, or any other choice of $b_1, \ldots, b_4$ whose sum is *not* zero.

Mastering Linear Algebra Concepts: Span

Please begin by reviewing "How to Study Linear Algebra," at the beginning of this *Study Guide*.

To really understand a key concept, you need to form an image in your mind that consists of the basic definition(s) together with many related ideas. Your goal at this point is to collect various ideas associated with the set Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and the concept of a set that "spans" $\mathbb{R}^n$. Here are specific things to do now as you prepare a sheet (or sheets) for review and reference.

- Write the **definition** of Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. (Learn it word for word.)
- Write the **definition** of the phrase: $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ spans $\mathbb{R}^n$. (See page 43.) Here *span* is a verb rather than a noun as in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.
- Add the **equivalent description** (not definition) of what is meant for a vector **b** to be in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. (See page 35.)
- Copy **Theorem 4** word for word. (If you try to rephrase or summarize it in your own words, you are likely to change the meaning.)

- Sketch some **geometric interpretations** of Span{$\mathbf{v}_1, \ldots, \mathbf{v}_p$}. (Select some of Figs. 8, 10, 11, and Exercises 19, 20 in Section 1.3.)
- Identify **special cases**. (Describe Span{**u**} and Span{**u, v**}.
- Summarize **algorithms** or **typical computations** (such as Example 6 and Exercises 11–14, 17, 18, 25, and 26 in Section 1.3, or Example 3 and Exercises 13–22 in Section 1.4.
- Describe connections with other concepts. (See pages 42–43.)

Whenever you encounter new examples or situations that help you understand the concept of a spanning set, add them to this review sheet.

MATLAB `gauss` **and** `bgauss`

To solve **`Ax = b`**, row reduce the matrix **`M = [A b]`**. The command **`x = [5;3; –7]`** creates a column vector **x** with entries 5, 3, –7. Matrixvector multiplication is **`A*x`**.

To speed up row reduction of **`M = [A b]`**, the command **`gauss(M,r)`** will use the leading entry in row *r* of *M* as a pivot, and use row replacements to create zeros in the pivot column below this pivot entry. The result is stored in the default matrix “ans”, unless you assign the result to some other variable, such as *M* itself.

For the backward phase of row reduction, use **`bgauss(M,r)`**, which selects the leading entry in row *r* of *M* as the pivot, and creates zeros in the column *above* the pivot. Use **`scale`** to create leading 1’s in the pivot positions. The commands **`gauss`**, **`bgauss`**, and **`scale`** are in the Laydata Toolbox, which you can download from the web.

Visualize the solution set of a homogeneous equation $A\mathbf{x} = \mathbf{0}$ as:

- the single point $\mathbf{0}$, when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution,
- a line through $\mathbf{0}$, when $A\mathbf{x} = \mathbf{0}$ has one free variable,
- a plane through $\mathbf{0}$, when $A\mathbf{x} = \mathbf{0}$ has two free variables.
  (For more than two free variables, also use a plane through $\mathbf{0}$.)