4.2 Null Spaces, Column Spaces, & Linear Transformations

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that $\mathbf{0}$ is in Nul A. Since __________, $\mathbf{0}$ is in

__________.

Property (b) If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A, show that $\mathbf{u} + \mathbf{v}$ is in Nul A. Since $\mathbf{u}$ and $\mathbf{v}$ are in Nul A,

__________ and __________.

Therefore

$A(\mathbf{u} + \mathbf{v}) =$ ________ $+$ ________ $=$ _______ $+$ _______ $=$ ____.

Property (c) If **u** is in Nul A and c is a scalar, show that $c\mathbf{u}$ in Nul A:

$$A(c\mathbf{u}) = \underline{\quad\quad} A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.

Solving $A\mathbf{x} = \mathbf{0}$ yields an ***explicit description*** *of Nul A.*

EXAMPLE: Find an explicit description of Nul A where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$$

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \ x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + \ x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\text{Nul } A = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

Observations:

1. Spanning set of Nul A, found using the method in the last example, is automatically linearly independent:

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$c_1 =$ ______ $\qquad c_2 =$ ______ $\qquad c_3 =$ ______

2. If Nul A$\neq \{\mathbf{0}\}$, the the number of vectors in the spanning set for Nul A equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

The **column space** of an $m \times n$ matrix A (Col A) is the set of all linear combinations of the columns of A.

If $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$, then

$$\boxed{\text{Col } A = \text{Span}\{\mathbf{a}_1, \ \dots, \ \mathbf{a}_n\}}$$

THEOREM 3

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

Why? (Theorem 1, page 221)

Recall that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the columns of A. Therefore

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}}$$

EXAMPLE: Find a matrix A such that $W = Col\ A$ where

$$W = \left\{ \begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} : x, y \text{ in } \mathbf{R} \right\}.$$

Solution:

$$\begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{Therefore } A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

By Theorem 4 (Chapter 1),

> The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$.

The Contrast Between Nul A and Col A

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(b) The null space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(c) Find a nonzero vector in Col A. (There are infinitely many possibilities.)

$$\_\_\_\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \_\_\_\begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \_\_\_\begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

(d) Find a nonzero vector in Nul A. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -2x_2 \\ x_2 & \text{ is free} \\ x_3 &= 0 \end{aligned}$$

Let $x_2 =$ ___ and then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Contrast Between Nul A and Col A where A is $m \times n$ (see page 232)

Review

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each **u** and **v** in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

THEOREM 1, 2 and 3 (Sections 4.1 & 4.2)

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

EXAMPLE: Determine whether each of the following sets is a vector space or provide a counterexample.

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$

Solution: *Since* $\_\_\_\_\_ = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$ is not in H, H is not a vector space.

(b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x - y = 0 \\ y + z = 0 \end{array} \right\}$

Solution: Rewrite $\begin{array}{l} x - y = 0 \\ y + z = 0 \end{array}$ as

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $V =$Nul A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since Nul A is a subspace of $\mathbf{R}^2$, V is a vector space.

(c) $S = \left\{ \begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$

One *Solution:* Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$; therefore S is a vector space by Theorem 1.

Another Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$S = \text{Col } A$ where $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}$; therefore S is a vector space, since a column space is a vector space.

Kernal and Range of a Linear Transformation

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u},\mathbf{v}$ in V;

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in tin V and all scalars c.

The ***kernel*** (or **null space**) of T is the set of all vectors $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$. The ***range*** of T is the set of all vectors in W of the form $T(\mathbf{u})$ where $\mathbf{u}$ is in V.

So if $T(\mathbf{x}) = A\mathbf{x}$, col A = range of T.