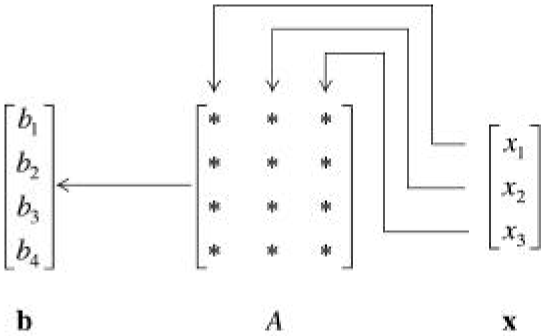
1.8 INTRODUCTION TO LINEAR TRANSFORMATIONS

Linear transformations are important for both the theory and the applications of linear algebra. You will see both uses in a variety of settings throughout the text. The graphical descriptions in this section will be augmented in Section l.9 and in a later section on computer graphics.

STUDY NOTES

Viewing the correspondence from a vector $\mathbf{x}$ to a vector $A\mathbf{x}$ as a mapping provides a dynamic interpretation of matrix-vector multiplication and a new way to understand the equation $A\mathbf{x} = \mathbf{b}$. Using the language of computer science, we can describe a matrix in two ways—as a data structure (a rectangular array of numbers) and as a program (a prescription for transforming vectors). Strictly speaking, however, the actual linear transformation is the function or mapping $\mathbf{x} \mapsto A\mathbf{x}$ rather than just A itself.

Here is a way to visualize a matrix acting as a linear transformation. The entries in the input vector **x** are assigned as weights that multiply the corresponding columns of A, then the resulting weighted columns are added together to produce the output vector **b**.



As you learn the definition of a linear transformation T, don't forget the crucial phrases "for all **u** and **v** in the domain of T" and "for all **u** and all scalars c." The mapping T defined by $T(x_1, x_2) = (|x_2|, |x_1|)$ is *not* a linear mapping, and yet T satisfies the linearity properties for *some* vectors in its domain and *some* scalars.

The key exercises are 17–20, 25 and 31.

SOLUTIONS TO EXERCISES

1. $T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$

5. $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

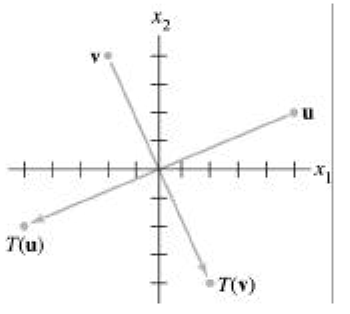
Note that a solution is *not* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. To avoid this common error, write the equations:

$$\begin{aligned} x_1 \quad + 3x_3 &= 3 \\ x_2 + 2x_3 &= 1 \end{aligned}$$ and solve for the basic variables: $$\begin{cases} x_1 = 3 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_3 \text{ is free} \end{cases}$$

General solution: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-3x_3 \\ 1-2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$. For a particular solution, one might choose $x_3 = 0$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

7. $a = 5$; the domain of T is $\mathbb{R}^5$, because a 6$\times$5 matrix has 5 columns and for $A\mathbf{x}$ to be defined, $\mathbf{x}$ must be in $\mathbb{R}^5$. $b = 6$; the codomain of T is $\mathbb{R}^6$, because $A\mathbf{x}$ is a linear combination of the columns of A, and each column of A is in $\mathbb{R}^6$.

13. The transformation may be described geometrically as a reflection through the origin. Two other correct descriptions are a rotation of π radians about the origin and a rotation of $-\pi$ radians about the origin. See the figure.



18. *Additional Hint*: Draw a line through $\mathbf{w}$ parallel to $\mathbf{v}$, and draw a line through $\mathbf{w}$ parallel to $\mathbf{u}$. This will help you write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

19. All you know are the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ and the fact that T is linear. The key idea is to write

$\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\mathbf{e}_1 - 3\mathbf{e}_2$. Then, from the linearity of T, write

$$T(\mathbf{x}) = T(5\mathbf{e}_1 - 3\mathbf{e}_2) = 5T(\mathbf{e}_1) - 3T(\mathbf{e}_2) = 5\mathbf{y}_1 - 3\mathbf{y}_2 = 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}$$

To find the image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, observe that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) = x_1\begin{bmatrix}2\\5\end{bmatrix} + x_2\begin{bmatrix}-1\\6\end{bmatrix} = \begin{bmatrix}2x_1 - x_2\\5x_1 + 6x_2\end{bmatrix}$$

21. **a**. A function is another word for transformation or mapping.

b. See the paragraph before Example 1.

c. See Figure 2. Or, see the paragraph before Example 1.

d. See the paragraph after the definition of a linear transformation.

e. See the paragraph following the box that contains equation (4).

25. Any point **x** on the line through **p** in the direction of **v** satisfies the parametric equation **x** = **p** + t**v** for some value of t. By linearity, the image T(**x**) satisfies the parametric equation

$$T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{v}) = T(\mathbf{p}) + tT(\mathbf{v}) \qquad (*)$$

If T(**v**) = **0**, then T(**x**) = T(**p**) for all values of t, and the image of the original line is just a single point. Otherwise, (*) is the parametric equation of a line through T(**p**) in the direction of T(**v**).

Study Tip: Exercise 31 is important, because it will help you to connect the concepts of linear dependence and linear transformation. Be sure to try the exercise first, before looking in the answer section of the text. Don't feel badly if you need to peek at the hint there. Only my best students can do this problem unaided. Once you have seen the hint, try hard to construct the desired explanation without consulting the solution I have written below. Don't give up too soon. Reread the definitions of linear dependence and linear transformation, if necessary.

After you have written your best attempt at an explanation, check it against the *Study Guide* solution. Also, study the strategy there of how I found the solution. Even if your attempt is quite unsatisfactory, the time spent on this problem is worthwhile, because you will learn more from the solution here.

31. To help you use this *Study Guide* properly, I have hidden the solution at the end of the solutions for Section 1.9. *Do not look there until you have followed the instructions above*. (I may not "hide" a solution again, but I wanted this one time to emphasize the importance of working seriously on a problem before checking the solution.)

Mastering Linear Algebra Concepts: Linear Transformation

Start to form a robust mental image of a linear transformation by preparing a review sheet that covers the following categories:

• definition	Page 77
• equivalent descriptions	Equations (4) and (5)
• geometric interpretations	Figs. 1 or 2
• special cases	Matrix transformation: page 77
• examples and "counterexamples"	Superposition, Examples 2-6 Paragraph before Exercise 1, in this *Guide* Exercises 29, 30, 33
• connections with other concepts	Existence and uniqueness: page 75 Linear dependence: Exercise 31

Note: Exercise 31 should enrich your mental image of linear dependence, so add a note about it to your list for "linear independence". If your course does not emphasize the next section, turn now to the end of the *Study Guide* material for Section 1.9 and read the box on *Existence and Uniqueness*.