1.9 MATRIX OF A LINEAR TRANSFORMATION

Every matrix transformation is a linear transformation. This section shows that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation. Chapters 4 and 5 will discuss other examples of linear transformations.

KEY IDEAS

A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is completely determined by what it does to the columns of the identity matrix I_n. The jth column of the standard matrix for T is $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of I_n.

There are two ways to compute the standard matrix A. Either compute $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$, which is easy to do when T is described geometrically, as in Exercises 1–14, or fill in the entries of A by inspection, which is easy to do when T is described by a formula, as in Exercises 15–22.

Existence and uniqueness questions about the mapping $\mathbf{x} \mapsto A\mathbf{x}$ are determined by properties of A. You should know how this works. The proof of Theorem 11 also applies to linear transformations on the general vector spaces in Chapter 4. Here is another way to understand Theorem 11 [and Theorem 12(b)] using the language of matrix transformations:

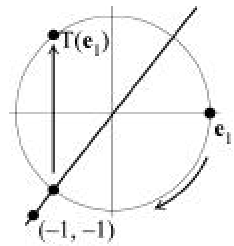
> Let A be the standard matrix of T. Then T is one-to-one if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$. This happens if and only if every column of A is a pivot column, which happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

The "if and only if" phrase in Theorem 11 (and in the proof above) was discussed in *A Mathematical Note*, in Section 1.2 of this *Guide*.

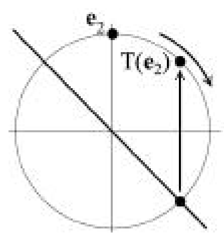
SOLUTIONS TO EXERCISES

1. The columns of the standard matrix A of T are the images of $\mathbf{e}_1$ and $\mathbf{e}_2$. Write these images *vertically*: $T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix}$. Then $A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$.

7. Follow what happens to $\mathbf{e}_1$ and $\mathbf{e}_2$. Since $\mathbf{e}_1$ is on the unit circle in the plane, it rotates through $-3\pi/4$ radians into a point on the unit circle that lies in the third quadrant and on the line $x_2 = x_1$ (that is, $y = x$ in more familiar notation). The point $(-1, -1)$ is on the line $x_2 = x_1$, but its distance from the origin is $\sqrt{2}$. So the rotational image of $\mathbf{e}_1$ is $(-1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(-1/\sqrt{2}, 1/\sqrt{2})$.



Similarly, $\mathbf{e}_2$ rotates into a point on the unit circle that lies in the second quadrant and on the line $x_2 = -x_1$, namely, $(1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(1/\sqrt{2}, 1/\sqrt{2})$.



These are UNCORRECTED page proofs.

When the two calculations described above are written in vertical vector notation, the transformation's standard matrix $[T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)]$ is easily seen:

$$\mathbf{e}_1 \to \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{e}_2 \to \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Checkpoint: Use an idea from this section to explain why the linear transformation T that reflects points through the origin, $T(x_1, x_2) = (-x_1, -x_2)$, is the same as the linear transformation R that rotates points about the origin in $\mathbb{R}^2$ through π radians.

13. Since $(2, 1) = 2\mathbf{e}_1 + \mathbf{e}_2$, the image of (2, 1) under T is $2T(\mathbf{e}_1) + T(\mathbf{e}_2)$, by linearity of T. On the figure in the exercise, locate $2T(\mathbf{e}_1)$ and use it with $T(\mathbf{e}_2)$ to form the parallelogram shown in the text's answers.

19. The matrix A that changes (x_1, x_2, x_3) into $(x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$ can be found by inspection when vectors are written in column formation. Write a blank matrix A to the left of the column vector $\mathbf{x}$ and fill in the entries of A. Since $T(\mathbf{x})$ has 2 entries, A has 2 rows. Since $\mathbf{x}$ has 3 entries, A must have 3 columns.

$$\begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

21. $T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To solve $T(\mathbf{x}) = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, row reduce the augmented matrix: $\begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$.

Study Tip: When T is described by a formula, as in Exercises 15–22, you can use the method of Exercise 19 to find an A such that $T(\mathbf{x}) = A\mathbf{x}$, *provided* that T is a linear transformation. (Finding A *proves* that T is linear.) If you can't find the matrix, T is probably *not* a linear transformation. To show that such a T not linear, you have either to find two vectors $\mathbf{u}$ and $\mathbf{v}$ such that $T(\mathbf{u} + \mathbf{v})$ is not equal to $T(\mathbf{u}) + T(\mathbf{v})$ or to find a vector $\mathbf{u}$ and scalar c such that $T(c\mathbf{u}) \neq cT(\mathbf{u})$.

The text does not give you practice determining whether a transformation is linear because the time needed to develop this skill would have to be taken away from some other topic. If you are expected to have this skill, you will need some exercises (besides Exercises 32 and 33 in Section 1.8). Check with your instructor.

23. **a**. See Theorem 10.

b. See Example 3.

c. See the paragraph before Table 1.

d. See the definition of *onto*. *Any* function from $\mathbb{R}^n$ to $\mathbb{R}^m$ maps each vector onto another vector.

e. See Example 5.

25. Three row interchanges on the standard matrix A of the transformation T in Exercise 17 produce $\begin{bmatrix} \textcircled{1} & 1 & 0 & 0 \\ 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This matrix shows that A has only three pivot positions, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one. Also, since A does not have a pivot in each row, the columns of A do not span $\mathbb{R}^4$. By Theorem 12, T does *not* map $\mathbb{R}^4$ onto $\mathbb{R}^4$.

31. *T is one-to-one if and only if* A *has* n *pivot columns*. This statement follows by combining Theorem 12(b) with the statement in Exercise 30 of Section 1.7.

A Mathematical Note: One-to-one

Many students have difficulty with the concept of a one-to-one mapping. Figure 4 should help. The transformation T on the left appears to map three (or even more) points to one image point. In contrast, the transformation T on the right maps three points to one image point. In contrast, the transformation T on the right maps three points to three points. You could say that T is three-to-three (or six-to-six), but the standard terminology is one-to-one.

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33. Define $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(\mathbf{x}) = B\mathbf{x}$ for some m 3 n matrix B, and let A be the standard matrix for T. By definition, $A = [T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$, where $\mathbf{e}_j$ is the jth column of I_n. However, by matrix-vector multiplication,

$T(\mathbf{e}_j) = B\mathbf{e}_j = \mathbf{b}_j$, the jth column of B. So $A = [\mathbf{b}_1 \cdots \mathbf{b}_n] = B$.

35. If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then its standard matrix A has a pivot in each row, by Theorem 12 and by Theorem 4 in Section 1.4. So A must have at least as many columns as rows, so $m \quad n$.

When T is one-to-one, A must have a pivot in each column, by Theorem 12, so m n.

37. [M] There is no pivot in the fourth column, so the columns of the matrix are not linearly independent and hence the linear transformation is not one-to-one (Theorem 12). (Or, use the result of Exercise 31.)

39. [M] Row reduction of the matrix shows that columns 1, 2, 3, and 5 contain pivots, but there is no pivot in the fifth row, so the columns of the matrix do not span $\mathbb{R}^5$. By Theorem 12, the linear transformation is not onto.

31. *(This solution is for Section 1.8.) To construct the proof, first write in mathematical terms what is given.*

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0 \qquad (*)$$

Next, think about what you must prove. In this problem, to prove that the image points are linearly dependent, you need a dependence relation among $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, and $T(\mathbf{v}_3)$. That fact suggests the next step.

Apply T to both sides of (*) and use linearity of T, obtaining

$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(0)$

and

$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = 0$

Since not all the weights are zero, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent set. This completes the proof.

Study Tip: Analyze the strategy above for solving Exercise 31 (in Section 1.8). This approach will work later in a variety of situations.

Answer to Checkpoint: The reflection T has the property that $T(\mathbf{e}_1) = -\mathbf{e}_1$ and $T(\mathbf{e}_2) = -\mathbf{e}_2$, while the rotation R has the property that $R(\mathbf{e}_1) = -\mathbf{e}_1$ and $R(\mathbf{e}_2) = -\mathbf{e}_2$. Since a linear transformation is completely determined by what it does to the columns $\mathbf{e}_1$ and $\mathbf{e}_2$ of the identity matrix, T and R must be the same transformation. (You could also explain this by observing that T and R have the same standard matrix, namely, $[-\mathbf{e}_1 \quad -\mathbf{e}_2]$.)

Mastering Linear Algebra Concepts: Existence and Uniqueness

It's time to review and organize what you have learned about existence and uniqueness concepts, if you have not already done so. The review will help to prepare you for an exam on the chapter material.

Search through the chapter and collect all the various ways to express existence and uniqueness statements. Most of them can be found in boxes (and theorems) with an "if and only if" statement. Also, check the exercises. For existence, make two lists—one that concerns the equation $A\mathbf{x} = \mathbf{b}$ for some fixed $\mathbf{b}$ (but not always phrased as a matrix equation), and one that concerns the existence of solutions of $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$.